2018 PiMC First Round

AlphaStar Academy

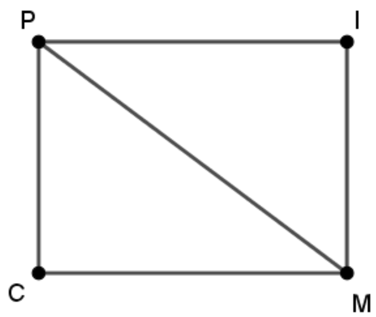
SOLUTIONS

1. (Handong Wang) Evaluate $7 - 3 \times 2$.

   Answer (1): Using the order of operations we get

   $$7 - 3 \times 2 = 7 - (3 \times 2) = 7 - 6 = \boxed{1}.$$

2. (Mehmet Kaysi) In the figure below, what is the ratio of the area of the rectangle $PIMC$ to the area of the triangle PMC?



   Answer (2): Note the the triangles PMC and PMI are identical, so their areas are equal. So, the area of the rectangle $PIMC$ is $\boxed{2}$ times the area of PMC.

3. (Stan Zhang) How many positive integers less than 100 are divisible by 13?

   Answer (7): The multiples of 13 less than 100 are $13 \times 1, 13 \times 2, \ldots, 13 \times 7 = 91$. So, there are $\boxed{7}$ of them.

4. (Iliya Shadfar)

   $$\frac{1}{3} + \frac{1}{1 + \frac{1}{2}} = ?$$

   Answer (1): We know $1 + \frac{1}{2} = \frac{3}{2}$ and $\frac{1}{3/2} = \frac{2}{3}$ then $\frac{2+1}{3} = \boxed{1}$.

5. (Eugene Chen) How many prime numbers are less than 20?

   Answer (8): 2, 3, 5, 7, 11, 13, 17, and 19 are the $\boxed{8}$ primes less than 20.

6. (Ayush Agarwal) Find the number of 6 letter sequences of A's and B's such that no two consecutive letters are the same.

   Answer (2): One can see that the only possibility is when the letters are alternating. Hence, the only $\boxed{2}$ possible sequences are $ABABAB$ and $BABABA$.

7. (Tomas Choi) A positive integer is called a *perfect number* if the sum of its divisors, including the number itself, is equal to twice the number. For example, 28 is a perfect number because the sum of its divisors is $1+2+4+7+14+28 = 56 = 2 \times 28$. What is the smallest perfect number?

   Answer (6): The sum of the factors of the first 6 numbers are shown below:

   1: 1
   2: $1 + 2 = 3$
   3: $1 + 3 = 4$
   4: $1 + 2 + 4 = 7$
   5: $1 + 5 = 6$
   6: $1 + 2 + 3 + 6 = 12$

   The first perfect number is $\boxed{6}$, since the sum of the factors is 12 which is twice 6.

8. (Iliya Shadfar) The ratio of the areas of two squares is 2 to 3. If the area of the smaller square is 6, what is the side length of the larger square?

   Answer (3): Note that $2/3 = 6/9$. So the area of the larger square is 9. Hence, its side length is 3.

9. (Stanley Wang) Alice is doing quick maths. She is able to do 13 calculations per second. Her friend, Bob, joins her, doing 778 calculations per minute. After two minutes, how many more calculations has Alice done compared to the number of calculations Bob has done?

   Answer (4): Alice does $13 \times 60 = 780$ calculations per minute. So, she does 2 more calculations per minute than Bob. After two minutes, Alice will have done $2 \times 2 = \boxed{4}$ more calculations.

10. (Stanley Wang) A worker earns \$1 on his first day, and on each of the following days, his pay is double the amount he earned the previous day. After several days, the worker saw that the total amount of money he earned was \$63. How many days did he work?

    Answer (6): Note that $63 = 1 + 2 + 4 + 8 + 16 + 32$. So the worker has worked for $\boxed{6}$ days.

Alternatively, notice that the cumulative amount of money the worker has accumulated up to a day is \$1 less than the amount of money he would earn the next day. For example, the worker earns $1 + 2 + 4 = 7$ dollars during first 3 days and earns 8 dollars on day 4. Since $63 = 2^6 - 1$, it has been $\boxed{6}$ days since the worker started working.

11. (Mehmet Kaysi) π is approximately 3.1415926535. After the decimal point, the first how many digits of π and $\frac{22}{7}$ are equal?

    Answer (2): By long division, note that 22/7 starts with 3.142. After the decimal point, only the first two digits (1 and 4) coincide with those of π. So, the answer is $\boxed{2}$.

12. (Stanley Wang) How many positive integers less than 40 are divisible by 2 and 5 but not 3?

    Answer (2): Note that multiples of 2 and 5 are just multiples of 10. So, the candidates are 10, 20, and 30. Among these only $\boxed{2}$ of them (10 and 20) are not multiples of 3.

13. (Bhanu Garg) Cindy takes five English tests during the school year, each scored out of 10 points. So far, Cindy has taken four tests and scored 5, 7, 9, and 10. If Cindy wants to have an average score of at least 8 on her five tests, what is the lowest she needs to score on her fifth test?

    Answer (9): Cindy needs an average of 8 over her five test scores, or a total of $8 \times 5 = 40$ as the sum. Since she scored $5 + 7 + 9 + 10 = 31$ so far, she needs at least $40 - 31 = 9$ on the fifth test.

14. (Brandon Wang) Tristan is taking a Math test with Algebra and Geometry sections. Tristan answered 48 of 75 Algebra questions correct and none of the Geometry questions correct. If he received a 60% on the Math test, how many Geometry questions are there on the test?

    Answer (5): Tristan got 60% on the test with 48 correct. So, 10% of the test is 8 questions and the whole test has 80 questions. Since 75 of the questions are Algebra, it leaves $\boxed{5}$ Geometry questions.

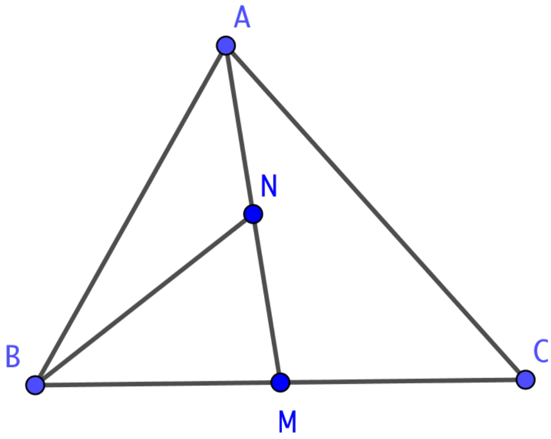
15. (Handong Wang) Find the units (ones) digit of $3^7 + 7^3$.

    Answer (0): 3^7 is equal to 2187, and 7^3 is equal to 343. Their sum is 2530 whose units digit is $\boxed{0}$.

    Alternatively, one can also note that the units digits of powers of 3 are $3, 9, 7, 1, 3, 9, 7$.

So, 3^7 ends in 7. Also, powers of 7 end in $7, 9, 3$ so 7^3 ends in 3. Finally, $7+3$ ends in 0.

16. (Jacob Nie) Let M be the midpoint of side BC on triangle ABC. Let N be the midpoint of AM. What is the ratio of the area of $\triangle ABC$ to the area of $\triangle BNM$?



Answer (4): Since M is the midpoint of side BC, $\triangle ABM$ and $\triangle AMC$ have equal areas (they have the same base and height lengths). Similarly, $\triangle ABN$ and $\triangle BNM$ have the same areas. The area of $\triangle BNM$ is half of the area of $\triangle ABM$ which is half the area of $\triangle ABC$. So, $\triangle ABM$ has area a quarter of the area of $\triangle ABC$. Hence, the answer is $\boxed{4}$.

17. (Jacob Nie) Jonathan the penguin is jumping along the number line. He starts at zero. Every time, he will jump either one unit to the right or one unit to the left. After seven jumps, at how many numbers could he possibly end up at?

Answer (8): Notice that Jonathan will never end up at an even number. Since, if he jumps to the right an even number of times, he must jump to the left an odd number of times, and vice versa. The points he can land on after five jumps are $-7, -5, -3, -1, 1, 3, 5$, and 7. That means there are $\boxed{8}$ total points he is able to land on.

18. (Mehmet Kaysi) What is the remainder when 201820182018 is divided by 9?

Answer (6): The remainder when a number is divided by 9 is equal to the remainder when the sum of its digits is divided by 9. The sum of the digits of the number is

$$3 \times (2 + 0 + 1 + 8) = 3 \times 11 = 33$$

whose remainder when divided by 9 is $\boxed{6}$.

19. (Ali Ersoz) In how many ways can 6 be written as a sum of three positive integers where the order of the integers does not matter? For example, 1+1+4 and 1+4+1 are considered the same.

Answer (3): Note that the smallest of the three positive integers must be either 1 or 2. If it is 1, there are two possibilities: $1 + 1 + 4$ or $1 + 2 + 3$. If it is 2, there is only one possibility: $2 + 2 + 2$. Hence, the answer is $\boxed{3}$.

20. (Stan Zhang) The start and end of a trail are 1 mile apart. Tom and Jerry both start at the beginning of the trail. Tom walks at a constant speed of 2 miles per hour and Jerry walks at a constant speed of 6 miles per hour. When Jerry reaches the end of the trail, he turns around and walks back until he reaches Tom; then he turns back and walks toward the end of the trail again. He repeats this until Tom has reached the end of the trail. How many miles does Jerry walk in total?

    Answer (3): Note that Jerry walks 3 times as fast as Tom. So during the same time, as Tom walks 1 mile, Jerry will walk $\boxed{3}$ miles.

21. (Kevin Chang) Victor wants to split an 8×12 rectangle into identical squares with integer side lengths. What is the smallest number of squares that he can get?

    Answer (6): To have the least number of squares we want the side length of the square as large as possible. It needs to divide both 8 and 12. So the largest side length we can get is 4. Finally, there are

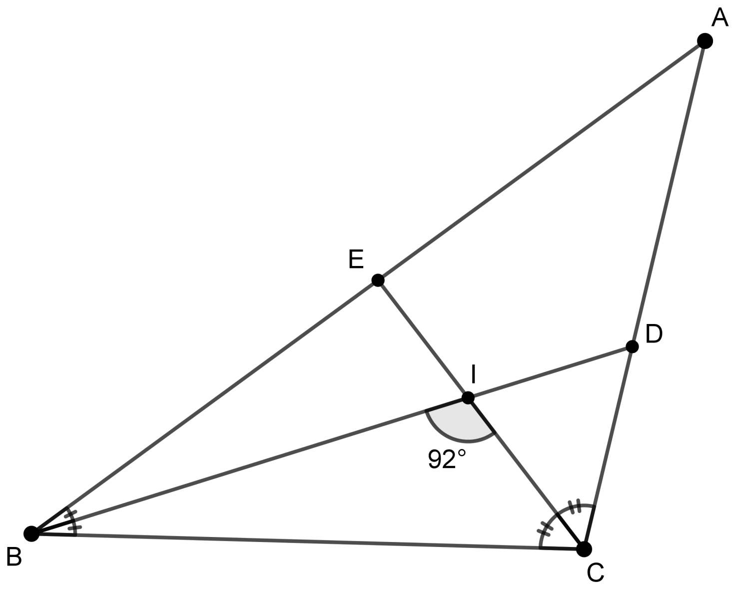
    $$\frac{8}{4} \times \frac{12}{4} = 2 \times 3 = \boxed{6}$$

    4×4 squares in an 8×12 rectangle.

22. (Kai-Siang Ang) Edwin knows that the password to his computer is one of 1,000 common passwords. His password is equally likely to be any of these passwords. So he decides to guess different passwords randomly at a constant rate. If the probability that Edwin can guess his password in 10 minutes is $\frac{1}{20}$, he is guessing at a rate of how many passwords per minute?

    Answer (5): In 10 minutes, Edwin has a $\frac{1}{20}$ chance of guessing his password. So, he must have guessed $\frac{1}{20}^{th}$ of all the possible passwords, in 10 minutes. This is $\frac{1000}{20} = 50$ passwords. So, he is guessing at a rate of $\frac{50}{10} = \boxed{5}$ passwords per minute.

23. (Mehmet Kaysi) In the figure below, BD and CE are angle bisectors and they intersect at I. If $\angle BIC = 92°$, what is the measure of $\angle BAC$, in degrees?



Answer (4): In $\triangle BIC$, the sum of the angles is 180°. Since one of the angles is 92°, the sum of the other two angles $\angle IBC + \angle ICB$ is 88°. Note that since these are half the angles of $\angle B$ and $\angle C$, the sum $\angle B + \angle C$ is twice this which is $2 \times 88 = 176$ degrees. Finally, adding $\angle A$ to these two angles gives the sum of angles in $\triangle ABC$ or 180°. So, $\angle A$ is $\boxed{4}$ degrees.

24. (Mehmet Kaysi) An elementary school teacher chooses 4 students – Pam, Ian, Max, and Cai – to compete in a math contest. She notices that each student has at least one friend on the team and also all students have the same number of friends on the team. In how many ways is this possible? Note that friendship is always mutual. For example, if Pam is a friend of Ian, then Ian is a friend of Pam.

Answer (7): Notation: We will use A-B to mean that A and B are friends.
Note that since each student has at least one friend on the team, the number of friends that each student has can be 1, 2, or 3. We will consider cases based on this number.

Case 1: Everyone has 1 friend on the team.
There are 3 possibilities as below:
(1) P-I, M-C
(2) P-M, I-C
(3) P-C, I-M

Case 2: Everyone has 2 friends on the team.
There are 3 possibilities as below (each corresponding to one of the cases below where friends and not friends are switched):
(1) P-C-I-M-P
(2) P-I-M-C-P
(3) P-M-C-I-P

Case 3: Everyone has 3 friends on the team.
Since everyone knows each other, there is only 1 possibility in this case.

So, the answer is $3 + 3 + 1 = \boxed{7}$.

25. (Kai-Siang Ang) A group of boys build 5 squares, each with side length equal to the number of boys. A group of girls build 3 squares, each with side length equal to the number of girls. The total area of all 8 squares is 120. Find the total number of boys and girls.

    Answer (8): Note that, if there are at least 5 boys, each square will have area at least 25 and 5 of these will have a total area of more than 120. So the number of boys can be at most 4. Next, note that the girls are building 3 identical squares of integer side length and hence their total areas must be divisible by 3.

    If there are 4 boys, the 5 squares they build will have total area of $5 \times 16 = 80$, leaving 40 for the total area of squares built by girls, which is not a multiple of 3. Similarly, if there are 1, 2, or 3 boys, the areas of the squares built by boys total 5, 20, 45, respectively leaving 115, 100, and 75 for the total area of squares built by the girls. Only 75 among these (corresponding to 3 boys) is a multiple of 3 in which case, each of the three square built by girls would have area 25. Hence there are 3 boys and 5 girls and the answer is $\boxed{8}$.