2017 PiMC

Final Round - Team Test

SOLUTIONS

1. (Caleb Ji) Compute the sum $1+2-3+4+5-6+7+8-9+10+11-12$.

   Answer (18): $(1+2-3)+(4+5-6)+(7+8-9)+(10+11-12) = 0+3+6+9 = 18$.

2. (Saranesh Prembabu) Alice and Bob love cats. Together they have 27 cats. Alice has 5 more cats than Bob. How many cats does Bob have?

   Answer (11): Besides the 5 extra cats that Alice has, they have 22 cats between them equally shared so that is 11 cats each. So Bob has 11 cats.

3. (Crystal Su) What is the value of

$$\frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{99}{98} \times \frac{100}{99}?$$

   Answer (50): We notice that most of the numerators and denominators cancel, leaving us with

$$\frac{\cancel{3}}{2} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdots \frac{\cancel{99}}{\cancel{98}} \cdot \frac{100}{\cancel{99}} = \frac{100}{2} = 50.$$

4. (Andrew Lin) My favorite number is a positive integer. When I square my favorite number, then divide by 4, and then subtract 7, I get 9. What is my favorite number?

   Answer (8): Let us work backwards:
   Last number is 9. The number before is 7 more so it is 16. The number before that is 4 times so it is 64. So, the first number is 8.

5. (Kevin Zhang) If two apples and three oranges cost 13 cents and three apples and two oranges cost 12 cents, what is the cost of one apple and one orange?

   Answer (5): Since two apples and three oranges cost 13 cents and three apples and two oranges cost 12 cents, adding them up means that five apples and five oranges cost 25 cents. Therefore, one apple and one orange cost 5 cents.

6. (Brian Gu) What is the sum of the factors of 56?

   Answer (120): The factors of 56 are $1, 2, 4, 7, 8, 14, 28, 56$. The sum of these eight factors is 120. Note that all the divisors can be found in the expansion of

   $$(1+2+4+8)(1+7) = 1\cdot 1 + 2\cdot 1 + 4\cdot 1 + 8\cdot 1 + 1\cdot 7 + 2\cdot 7 + 4\cdot 7 + 8\cdot 7.$$

   So the sum of divisors is

   $$(1+2+4+8)(1+7) = 15\cdot 8 = 120.$$

7. (Richard Spence) The date January 17, 2017 has the property that when it is written in the standard mm/dd/yy format, 01/17/17, the product of the month and the day equals the last two digits of the year. How many dates in the years 2017, 2018, 2019 and 2020 have this property?

   Answer (12): The numbers representing day and month must multiply to 17, 18, 19, or 20. The month can be a number 1-12. So the solutions are

   $$\begin{aligned} 17 &= 1\times 17, \\ 18 &= 1\times 18 = 2\times 9 = 3\times 6 = 6\times 3 = 9\times 2, \\ 19 &= 1\times 19, \\ 20 &= 1\times 20 = 2\times 10 = 4\times 5 = 5\times 4 = 10\times 2. \end{aligned}$$

   There are 12 such dates.

8. (Stanley Wang) George picks a positive integer from 1 to 100. When George reverses the digits of his integer and subtracts it from his original number, he obtains a difference of 45. What is the largest number George could have picked?

   Answer (94): Suppose George picks the number XY (here, X and Y are digits). We have George's number as $10X+Y$, and when he reverses the digits, his number becomes $X+10Y$. The difference between these numbers is $9(X-Y) = 45$, so $X - Y = 5$. To maximize the number George picks, we maximize X, the tens digit, so that $X = 9$ and $Y = 4$. Thus, the greatest number George can pick is 94.

9. (Ali Gurel) What is

   $$(2000 + 2001 + 2002 + \cdots + 2099) - (1900 + 1901 + 1902 + \cdots + 1999)?$$

   Answer (10,000): Note that there are 100 terms in each sum and each term in the first sum is 100 more than the corresponding term on the second sum. It is 2000 is 100 more than 1900, 2001 is 100 more than 1901 etc. So the first sum is $100 \times 100 = 10,000$ more than the second sum.

10. (James Shi) How many positive integers less than 100 are not divisible by 4 or 7?

    Answer (64): There are 14 positive integers less than 100 that are divisible by 7. There are 24 positive integers less than 100 that are divisible by 4. 28, 56, and 84 are the only positive integers in both lists. So there are $14 + 24 - 3 = 35$ positive integers less than 100 that are divisible by 4 or 7. Since there are a total of 99 positive integers less than 100, there are $99 - 35 = 64$ positive integers less than 100 that are not divisible by 4 or 7.

11. (Stanley Wang) A rhombus has each side of length 5. If one diagonal has length 8, what is the area of the rhombus?

    Answer (24): All rhombi have diagonals as perpendicular bisectors of each other. We see that the rhombus is made up of 3-4-5 right triangles, with a total area of $(3 \cdot 4/2) \cdot 4 = 24$.

12. (Matthew Hase-Liu) Spike walks at 2 miles per hour to get to work. As soon he gets there, however, he realizes that he forgot his laptop, so he runs back at 6 miles per hour, and then skips back to work at 3 miles per hour. What is his average speed?

    Answer (3): Let the distance from his home to work be d. Then, the total time is $\frac{d}{2} + \frac{d}{6} + \frac{d}{3} = d$. The total distance is $3d$, so the average speed is $\frac{3d}{d} = 3$.

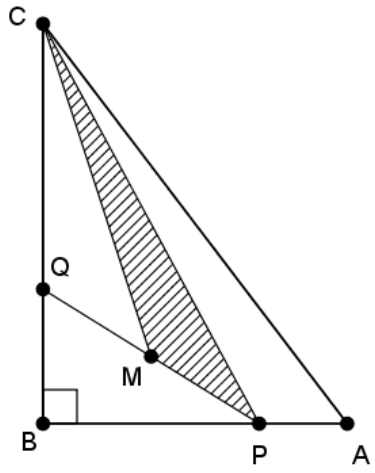
13. (Crystal Su) Bob is ordering a pizza for himself. He has a choice of 3 crusts, 2 types of cheese, and 5 toppings. He needs to pick 1 crust, 1 type of cheese, and at least 1 topping. How many ways can he order the pizza?

    Answer (186): He has 2 choices for each topping, either selecting it or not selecting it. So, there are $2 \times 2 \times 2 \times 2 \times 2 = 32$ ways to choose toppings. However, selecting none of the toppings is not allowed. Thus, he has 31 valid choices for the toppings. He also needs to choose one of 3 crusts, and one of 2 cheeses. So, we get a total of $31 \times 3 \times 2 = 186$ pizzas.

14. (Crystal Su) On Planet Math, a "beauty" is a positive integer less than 1000 that is divisible by 2, 3, and 5, but not 4. How many *beauties* exist on Planet Math?

    Answer (17): If a number is divisible by 2, 3 and 5, it must be divisible by $2 \times 3 \times 5 = 30$, because 2, 3, and 5 are all primes. There are 33 numbers less than 1000 that are divisible by 30, $30 \times 1, 30 \times 2, \ldots, 30 \times 33$. Among these, 16 are multiple of 4, $30 \times 2, 30 \times 4, \ldots, 30 \times 32$. The remaining 17 are not multiples of 4. Hence, the answer is 17.

15. (James Shi) $\triangle ABC$ is a right triangle with a right angle at vertex B. Points P and Q are on sides AB and BC, respectively. Point M is the midpoint of PQ. If $PB = 8$ and $QC = 4$, what is the area of $\triangle PMC$?



Answer (8): Triangle PQC has twice the area of triangle PMC, which can be shown by using line PQ as the base for both triangles. Both triangles have the same height and the base of triangle PQC is twice the base of triangle PMC. Changing the base of triangle PQC to QC, the area of triangle PQC is $\frac{bh}{2} = \frac{4 \cdot 8}{2} = 16$, so the area of triangle PMC is $\frac{16}{2} = 8$.