2017 PiMC

Final Round - Individual Test

SOLUTIONS

1. (Ali Gurel) $5 - 3 \times 2 + 1 = ?$

   Answer (0): $5 - 3 \times 2 + 1 = 5 - 6 + 1 = 5 + 1 - 6 = 0$.

2. (Evan Chen) Find the ones (units) digit of 5^{18}.

   $$5^{18} = \underbrace{5 \times 5 \times \cdots \times 5 \times 5}_{18 \text{ times}}.$$

   Answer (5): The powers of 5 all have ones (units) digit 5.

3. (Freya Edholm) At Poof School, 2 students disappear during each week of the school year. There were 43 students at Poof School at the beginning of the school year. After 17 full weeks, how many students have NOT yet disappeared from Poof School?

   Answer (9): Since 2 students disappear at the beginning of each week and 17 full weeks have passed, $2 \cdot 17 = 34$ students have disappeared since the beginning of the school year. There were 43 students initially at Poof School, so $43 - 34 = 9$ students have not yet disappeared.

4. (Brandon Wang) Kevin multiplies the first six positive integers. What is the smallest whole number that can be added to the product to get a multiple of 7?

   Answer (1): $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. The next multiple of 7 is 721. So we need to add 1 to get a multiple of 7. In general, this works for primes other than 7 as well. If you multiply all positive integers less than the prime, and add 1 to the product you get a multiple of the prime.

5. (Kevin Chang) What is the hundreds digit of $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$?

   Answer (3): Multiplying, we have $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320$. So the hundreds digit is 3.

6. (Kevin Chang) Nathan runs four times as fast as he walks. If it takes Nathan 6 minutes to walk 1 mile, how many minutes does it take him to run 2 miles?

   Answer (3): If Nathan walks 1 mile in 6 minutes, then he walks 2 miles in 12 minutes. Since he runs four times as fast, he runs 2 miles in 3 minutes.

7. (Edwin Peng) How many ways are there to choose 2 marbles from a bag of 4 different marbles, if the order of the selection is not important?

   Answer (6): There are 4 possible outcomes for the first choice, and 3 possible outcomes for the second choice. So the total number of possible chices is $4 \times 3 = 12$. However, we must divide by 2 since the selection order does not matter. Therefore the answer is $\frac{4 \times 3}{2} = 6$.

8. (Eugene Chen) The area of a circle with radius 6 is how many times the area of a circle with radius 2?

   Answer (9): The area of a circle is proportional to the square of its radius. So the answer is $6^2 \div 2^2 = 9$.

9. (Caleb Ji) $ABCD$ is a rectangle with perimeter 12. There is a magic number such that side AB has length 3 less than the magic number and side BC has length 1 less than the magic number. Find the area of rectangle $ABCD$.

   Answer (8): The perimeter of the rectangle is 12. So $AB + BC$ is 6. Note that since AB is 3 less and BC is 1 less than the magic number, AB and BC differ by 2. Since their sum is 6, we conclude that $AB = 2$ and $BC = 4$. The area of $ABCD$ is $2 \times 4 = 8$.

10. (Ashwath Thirumalai) The number 4 has exactly three positive factors, namely 1, 2, and 4. What is the smallest positive integer that has exactly four positive factors?

    Answer (6): Positive integers 1 through 5 have less than four factors. The number 6 however, has four factors, namely 1, 2, 3, and 6. Therefore, the answer is 6.

11. (Stanley Wang) Stanley selects a three digit positive integer and computes the sum of the digits. If Stanley obtains a sum of 25, how many three digit positive integers could Stanley have picked?

    Answer (6): There are 6 such numbers: 997, 979, 799, 988, 898, and 889.

12. (Ashwath Thirumalai) Let N be the smallest number greater than 1 that leaves a remainder of 1 when divided by 2, 3, 4, and 5. What is the sum of the digits of N?

    Answer (7): Note that $N - 1$ is a multiple of 2, 3, 4, and 5. The smallest multiple of 2, 3, 4, and 5 is 60. This can be seen by observing that any multiple of 2, 3, and 5 must be a multiple of $2 \times 3 \times 5 = 30$, since they are all primes. Among multiples of 30, the smallest one that is a multiple of 4 is 60. So the smallest N value greater than 1 is 61 whose sum of digits is $6 + 1 = 7$.

13. (Brandon Wang) Brandon is writing a sequence of numbers whose first term is 1 and second term is 2. Each term after the second term is either the sum of the two previous terms if the sum is odd or the previous term if the sum is even. For example, if two consecutive terms in the sequence were 11 and 13, their sum would be even. Therefore, the next term would be 13. What is the 2017th term of Brandon's sequence?

    Answer (5): Note that first several terms of the sequence are $1, 2, 3, 5, 5, \cdots$. We note that after this, all the terms of the sequence are 5's because $5 + 5$ is even. Thus 2017th term of the sequence is 5.

14. (Ashwath Thirumalai) How many ways are there to choose three people from a group of five people, namely Alex, Bob, Cindy, Diana, and Eric, to serve on a committee if Alex and Bob refuse to serve together?

    Answer (7): We will use three initials to represent each committee. For example ABC represents a committee of Alex, Bob, and Cindy. If neither Alex nor Bob is in the committee, then it must be CDE. If Alex is in the committee but not Bob, then the choices are ACD, ACE, and ADE. Similarly if Bob is in the committee but not Alex, then the choices are BCD, BCE, and CDE. Thus there are $1 + 3 + 3 = 7$ such committees.

15. (Kevin Chang) A 4×6 rectangle is split into three squares each one having an integer side length. Find the sum of these three lengths.

    Answer (8): Since there are only three squares, one of them must have a side of length 4, which is the smaller dimension of the rectangle. The combined area of the remaining two squares is $4 \times 6 - 4^2 = 8$. Given that they have integer side lengths, they must both have side length 2. Hence, the answer is $4 + 2 + 2 = 8$.
    Alternatively, the only way to write 24 as a sum of three positive squares is $2^2 + 2^2 + 4^2$. Hence, the answer is $2 + 2 + 4 = 8$.

16. (James Shi) If the area of a rectangle with integer side lengths is 36, how many distinct values are there for the perimeter of the rectangle?

    Answer (5): The possible rectangles are 1×36, 2×18, 3×12, 4×9, and 6×6. All of these rectangles have different perimeters. Therefore, the answer is 5.

17. (Richard Yi) Find the sum of all possible numbers that satisfy the following property: "36 divided by the number is equal to 24 minus three times the number".

    Answer (8): Let N be the number. Then we have $\dfrac{36}{N} = 24 - 3N$ or $\dfrac{36}{N} + 3N = 24$ which implies $N + \dfrac{12}{N} = 8$. So N and $12/N$ are two numbers with sum 8 and product 12. They must be 2 and 6 in some order. Therefore, the answer is $2 + 6 = 8$.

18. (Kevin Zhang) When you multiply three 7's together, you get $7 \times 7 \times 7 = 343$, which leaves a reminder of 2 when divided by 11. What is the smallest number of 7's you have to multiply together to get a remainder of 4 when divided by 11?

    Answer (6): Note that $7^6 = 7^3 \times 7^3 = (341 + 2) \times (341 + 2) = 11 \times A + 4$ leaves a remainder of 4 when divided by 11. On the other hand, $7^0 = 1$, $7^1 = 7$, $7^2 = 44 + 5$, and $7^3 = 341 + 2$ leave remainders of 1, 7, 5, and 2 when divided by 11. $7^4 = 7^2 \times 7^2 = (44 + 5) \times (44 + 5) = 11 \times B + 25$ leave a remainder of 3 when divided by 11. $7^5 = 7^2 \times 7^3 = (44+5) \times (341+2) = 11 \times C + 10$ leave a remainder of 10 when divided by 11. Therefore, the answer is 6.

19. (Andrew Lin) The square of 1 more than a positive number is equal to 17 more than twice the number. What is this number?

    Answer (4): Let N be the number. Then we have $(N + 1)^2 = N^2 + 2N + 1 = 2N + 17$ which simplifies to $N^2 = 16$. So $N = 4$.

20. (Eric Huang) A year is *mathy* if the sum of its digits is 10. For example, 2017 is a *mathy* year because $2 + 0 + 1 + 7 = 10$. How many *mathy* years are there in this century?

    Answer (9): We notice that any year from 2000 to 2099 has the first two digits of 20. That gives a sum of 2. Therefore the last two digits of these mathy years sum to 8. The last two digits have to be 08, 17, 26, 35, 44, 53, 62, 71, 80 for a total of 9 possibilities.

21. (Andrew Lin) Austin has 24 sticks, each 1 foot long. He uses them to build the largest possible cube. What is the volume of the cube, in cubic feet?

    Answer (8): There are 12 edges in a cube. So, each edge is $\frac{24}{12} = 2$ feet long. Therefore, the volume of the cube is $2^3 = 8$ cubic feet.

22. (Stan Zhang) What is the units digit of

    $$1^3 + 2^3 + 3^3 + \cdots + 19^3?$$

    Answer (0): $1^3 + 19^3$ has a units digit of 0, because 1^3 has a units digit of 1, and 9^3 has as units digit of 9. Similarly, $2^3 + 18^3$ has a units digit of 0 because 2^3 has a units digit of 8 and 18^3 has a units digit of 2. Continuing this way, every number pairs up with another number to form 0, except for 10^3, which has a units digit of 0.

23. (Handong Wang) Harry wants to buy gum for an 11-day trip. He can buy gum in two different sizes, 9 sticks per pack or 14 sticks per pack. During the trip, Harry will chew the same number of sticks of gum every day. What is the smallest number of packs of gum he needs to buy if he does not want any leftovers at the

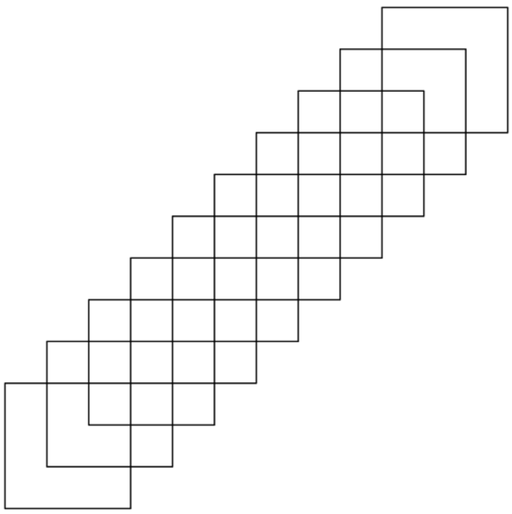
end of the trip?

Answer (5): The number of sticks of gum he buys in total must be a multiple of 11. Note that none of 11, 22, 33, or 44 can be expressed as a sum of 9's and 14's. On the other hand, $55 = 9 + 9 + 9 + 14 + 14$. Therefore, the answer is 5.

24. (Benjamin Chen) Pablo the Meerkat has 6 sticks of length 1, 2, 3, 4, 5, and 6. In how many ways can he select 3 of these sticks that form sides of a triangle with positive area, if the order of the selection of the 3 sticks is not important?

Answer (7): The sum of the two side lengths in a triangle must be larger than the third side length. We will do casework based on the length of the shortest stick which must be 2, 3, or 4. For 2, the triples are 2-3-4, 2-4-5, and 2-5-6. For 3, the triples are 3-4-5, 3-4-6, and 3-5-6. For 4, the only triple is 4-5-6. In total, there are 7 selections that work.

25. (Richard Spence) Ten overlapping 1×1 squares are placed as shown below, where side lengths of the squares are trisected equally. What is the total area covered by these ten squares?



Answer (6): Start with the bottom left square, which has area 1. Starting from bottom-left and working to top-right, every time we add a square, there is a $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ area overlap, so the total area increases by $1 - \frac{4}{9} = \frac{5}{9}$. Repeat this process for the remaining 9 squares, and the total area is $1 + \frac{5}{9} \times 9 = 6$.