PiMC 2016

TEAM ROUND

SOLUTIONS

1. (Alicia Weng) Calculate
$$\frac{20 \times 22 \times 24}{10 \times 11 \times 12}.$$

Answer (8):
$$\frac{20 \times 22 \times 24}{10 \times 11 \times 12} = \frac{20}{10} \times \frac{22}{11} \times \frac{24}{12} = 2 \times 2 \times 2 = 8.$$

2. (Ashwath Thirumalai) Mark and John start a 40-mile race together. Mark runs at the constant rate of 10 miles per hour and John runs at the constant rate of 5 miles per hour. How many hours does Mark wait at the finish line before John reaches there?

Answer (4): It will take Mark $40 \div 10 = 4$ hours to finish the race, and John $40 \div 5 = 8$ hours. So Mark will wait $8 - 4 = 4$ hours.

3. (Handong Wang) Harry has red, blue, and green marbles. The number of red marbles is prime and the number of blue marbles is composite. The number of green marbles is neither prime nor composite. What is the smallest number of marbles that Harry could have?

Answer (7): The smallest prime number is 2, the smallest composite number is 4. The only positive integer that is neither prime nor composite is 1. So Harry could have 2 red, 4 blue, and 1 green marble which gives a total of 7 marbles.

4. (Ali Gurel) The volume of a rectangular prism is the product of its length, width, and height. For example, if the length is 3, the width is 4, and the height is 5, then the volume of the prism is $3 \times 4 \times 5 = 60$. If you double the length, width, and height of a rectangular prism, by what percent does the volume increase?

Answer (700): The volume becomes $2 \times 2 \times 2 = 8$ times as before. So it increases by 700 percent.

5. (Evan Chen) Express $\frac{355}{113}$ as a decimal to the nearest hundred thousandth.

Answer (3.14159): Via long division, we get $\frac{355}{113} \approx 3.141592$. So the answer is

3.14159.

Remark: $\frac{355}{113}$ is the best approximation of π via fractions with denominators having at most 4 digits. The first 6 digits after the decimal point match with those of π.

6. (Alicia Weng) On the number line, what number, other than 1/5, is as far away from 1/6 as 1/5 is? Express your answer as a common fraction in simplest form.

   Answer (2/15): $\frac{1}{5}$ is $\left(\frac{1}{5}-\frac{1}{6}\right)=\frac{1}{30}$ more than $\frac{1}{6}$. So, the number we are looking for must be $\frac{1}{30}$ less than $\frac{1}{6}$.

   $$\frac{1}{6}-\frac{1}{30}=\frac{5}{30}-\frac{1}{30}=\frac{4}{30}=\frac{2}{15}.$$

7. (Ali Gurel) A triangle is isosceles if two of its angles have the same measure. In an isosceles triangle $\triangle ABC$, $\angle A = 50°$. What is the sum of the largest and smallest possible degree measures of $\angle B$?

   Note: You may use the fact that the sum of the measures of three angles of a triangle is 180 degrees.

   Answer (130): There are 3 possibilities regarding which two angles are same. If $\angle A = \angle B$ then $\angle B = 50$ degrees. Likewise, if $\angle A = \angle C$, then $\angle B = 80$ degrees. Finally, if $\angle B = \angle C$, then $\angle B = 65$ degrees. The largest of these is 80 degrees and the smallest is 50 degrees. So the answer is $80 + 50 = 130$.

8. (Benjamin Chen) Allen has 5 identical potatoes. In how many ways can Allen put these potatoes in 3 different boxes if each box must contain at least one potato?

   Answer (6): The possible distributions are: $(3,1,1)$, $(1,3,1)$, $(1,1,3)$, $(2,2,1)$, $(2,1,2)$, $(1,2,2)$. There are a total of 6 distributions.

   Alternatively, if he puts 1 potato in each box, he still needs to put 2 more. He can either put these 2 potatoes in the same box (in 3 ways) or in two different boxes (also in 3 ways). So there are 6 ways in total.

9. (Julia Huang) Four years ago, Julie was three times as old as Julia. Next year, Julia will be half as old as Julie. What is the sum of their ages now?

   Answer (28): If we let Julie's current age be e, and Julia's current age be a, then we can write the following two equations:

   $$e-4=3(a-4), \quad \text{and} \quad e+1=2(a+1).$$

Isolating e, we get

$$e = 3a - 8, \quad \text{and} \quad e = 2a + 1.$$

Since the two equations are equal, we can solve for a:

$$3a - 8 = 2a + 1.$$

We get $a = 9$. Then $e = 2a + 1 = 19$. The sum of their current ages is $e + a = 9 + 19 = 28$.

10. (Katherine Tian) Christie is making paper cranes while Katrina unfolds them. It takes Christie 2 minutes to make one paper crane and it takes Katrina 15 seconds to unfold one. Christie starts with 21 cranes and makes more while Katrina unfolds them. How many seconds will pass before Christie has no cranes left?

    Answer (345): In 2 minutes, or 120 seconds, Christie makes 1 crane, and Katrina unfolds $120 \div 15 = 8$ cranes. So there are $8 - 1 = 7$ fewer cranes after every 2 minutes. Christie starts with $21 = 7 \times 3$ cranes. After 4 minutes, she will have $21 - (7 \times 2) = 7$ cranes left. Katrina will unfold all of these 7 in 1 minute and 45 seconds at which time Christie will still be working on finishing her next paper crane. So after 5 minutes and 45 seconds (or after 345 seconds), Christie will have no cranes left.

11. (Katherine Tian) Claire has a basket of Easter eggs. If she divides the eggs into groups of 4, she has 3 eggs left. If she divides the eggs into groups of 5, she has 4 eggs left. If she divides the eggs into groups of 6, she has 5 eggs left. What is the smallest possible number of eggs in her basket?

    Answer (59): Claire needs one more egg to make groups of 4, 5, and 6. The smallest multiple of $4, 5$, and 6 is 60. So the answer is 1 less which is 59.

12. (Stanley Wang) Call an integer "exquisite" if it leaves the same remainder when divided by 3 and 4. How many exquisite integers are there 1 through 100?

    Answer (26): If an integer leaves a remainder of r when divided by 3 and a remainder of r also when divided by 4, with $0 \leq r \leq 2$, it must leave a remainder of r when divided by 12. Exquisite integers must then leave remainders 0, 1, or 2 when divided by 12. The exquisite integers from 1 to 100 inclusive can be grouped into sets: $\{1, 2\}, \{12, 13, 14\}, \{24, 25, 26\}, \ldots, \{96, 97, 98\}$. We have one set of 2 and 8 sets of 3 numbers, for a total of 26 exquisite integers.

13. (Kevin Chang) On a 4×4 chessboard, how many ways are there to place 2 identical rooks so that they do not attack one another? Rooks attack each other if they are placed in the same row or column.

    Answer (72): There are $4 \times 4 = 16$ possible positions to place the first rook. Since the first rook eliminates one column and one row, there are $3 \times 3 = 9$ possible positions for the second rook. This gives $16 \times 9 = 144$ ways. However the

two rooks are identical so each case is counted twice. Dividing by 2 we get 72 ways.

14. (Akshay Ravikumar) How many two-digit numbers are divisible by 1 more than their tens digit and 1 more than their ones digit?

    Answer (6): Let ab be the two-digit number. First, we notice that b cannot be odd, because then it cannot be divisible by an even number. If b is 0, then the number is a multiple of 10. Checking all multiples of 10, we have the numbers 10, 40 and 90. If b is 2, then the number is a multiple of 3. The only multiples of 3 that end in 2 are 12, 42, and 72. We note that 12 and 72 both work. If b is 4, then the number is a multiple of 5. This is not possible. If b is 6, then the number must be a multiple of 7. The only multiple of 7 that ends in 6 is 56 - however, this number does not work. If b is 8, then the number must be a multiple of 9. The only multiple of 9 that ends in 8 is 18, which works. Therefore, the possible values are $10, 12, 18, 40, 72$, and 90, and the answer is 6.

15. (Tomas Choi) Tomas reads a math book. He reads 1 page on the first day, 2 pages on the second day, 3 pages on the third day and so on, each day reading one more page than the previous day. If the book has 2016 pages, how many days will it take Tomas to finish the book?

    Answer (63): During the first 10 days, he will read $1 + 2 + 3 + \cdots + 10 = 55$ pages. During the second 10 days, he will read $10 \times 10 = 100$ pages more since on 11th day he will read 10 more pages than 1st day, on 12th day he will read 10 more pages than 2nd day etc. So he will read $55 + 100 = 155$ pages during the second 10 days, similarly $155 + 100 = 255$ pages during the third 10 days and so on. Keeping this up we note that he reads $55 + 155 + 255 + 355 + 455 + 555 = 1830$ pages during the first 60 days. After this, he will have $2016 - 1830 = 186$ more pages to read which takes him 3 more days since he reads 61, 62, and 63 pages on the next 3 days and $61+62+63 = 186$. So it takes him 63 days total to finish the book.