

PiMC 2016
INDIVIDUAL ROUND
SOLUTIONS

1. (Ali Gurel) What is 5% of 20?

Answer (1): 5% of a number is $1/20$ of the number. $1/20$ of 20 is 1.

2. (Ali Gurel) $2 - 2 \times 2 + 2 = ?$

Answer (0): Doing multiplication first and reordering terms, we get

$$2 - 4 + 2 = 2 + 2 - 4 = 4 - 4 = 0.$$

3. (James Shi) What is the tens digit of the product

$$3 \times 3 \times 3 \times 3 \times 3 \times 3?$$

Answer (2): By the associative property,

$$\begin{aligned} & 3 \times 3 \times 3 \times 3 \times 3 \times 3 \\ &= (3 \times 3) \times (3 \times 3) \times (3 \times 3) \\ &= 9 \times 9 \times 9 = (9 \times 9) \times 9 = 81 \times 9 = 729. \end{aligned}$$

The tens digit of 729 is 2.

4. (Ali Gurel) Two different regular 6-sided dice are rolled. Among 36 possible outcomes how many of them have the two values shown on top of the dice sum to 10?

Answer (3): The sum can be 10 in 3 different ways: $6 + 4$, $5 + 5$ or $4 + 6$.

5. (Ali Gurel) Simplify

$$8 \times \left(\frac{2}{6} + \frac{3}{24} + \frac{1}{24} \right).$$

Answer (4): First adding the last two fractions, we get

$$\begin{aligned} & 8 \times \left(\frac{2}{6} + \frac{3}{24} + \frac{1}{24} \right) \\ &= 8 \times \left(\frac{2}{6} + \frac{4}{24} \right) = 8 \times \left(\frac{2}{6} + \frac{1}{6} \right) \\ &= 8 \times \frac{3}{6} = 8 \times \frac{1}{2} = 4. \end{aligned}$$

6. (Crystal Su)

$$\begin{aligned}\star &= \square + \square \\ \square &= \diamond \times \diamond \\ \diamond &= \triangle \div \triangle\end{aligned}$$

In the equations above, each symbol represents a positive integer. What is the value of \star ?

Answer (2): We will work backwards. In the last equation: $\diamond = \triangle \div \triangle$, since any number divided by itself is 1, we conclude that $\diamond = 1$. Then, $\square = \diamond \times \diamond = 1 \times 1 = 1$. Finally, $\star = \square + \square = 1 + 1 = 2$.

7. (Evan Chen) There are 25 prime numbers less than or equal to 100. What is the ones (units) digit of their product?

Answer (0): Since 2 and 5 are two of these primes, the product is divisible by 10, and hence the units digit is zero.

8. (Caleb Ji) Danielle has two shirts, two skirts, and two hats. If an outfit consists of a shirt, a skirt, and a hat, how many different outfits can Danielle wear?

Answer (8): There are 2 choices for each article of clothing, giving a total of $2 \times 2 \times 2 = 8$ total outfits.

9. (Ashwath Thirumalai) What is 20% of 25% of 40% of 50?

Answer (1): 40% of 50 is 20. 25% of 20 is 5. 20% of 5 is 1.

10. (Daniel Yang) If x is a number such that $256^x = 4^{20}$, what is the value of x ?

Answer (5): Note that $256 = 4^4$. So, $256^x = (4^4)^x = 4^{4x} = 4^{20}$. Therefore, $4x = 20$ and $x = 5$.

11. (Eric Huang) Four more than double the number of dimes in a dollar is equal to the number of quarters in how many dollars?

Answer (6): There are 10 dimes in a dollar. Four more than double of 10 is $4 + 2 \times 10 = 4 + 20 = 24$. Finally, 24 quarters has a value of 6 dollars.

12. (Andrew Lin) Justin runs at the rate of 12 kilometers per hour. A mile is approximately 1.6 kilometers. At this rate, how many hours does it take Justin to run 30 miles? Round your answer to the nearest integer.

Answer (4): 30 miles is approximately $1.6 \times 30 = 48$ kilometers. In 1 hour, Justin runs 12 kilometers. So it takes him approximately $48 \div 12 = 4$ hours.

13. (Alicia Weng) Danielle and Rachel have an equal number of berries. After Danielle eats two of her own berries, Celine comes and takes two-thirds of Danielle's remaining berries, and Rachel eats 80% of her own berries. At the end, Danielle and Rachel have the same number of berries again. How many berries did Danielle have at the beginning?

Answer (5): If Danielle and Rachel each have n berries to begin with, then at the end they have $\frac{n-2}{3}$ and $\frac{n}{5}$ berries left, respectively. Setting these equal gives $n = 5$.

14. (Caleb Ji) The area and the perimeter of a rectangle are numerically equal. If the length of the rectangle is 6, what is the width of this rectangle?

Answer (3): Letting the width of the rectangle be w , the area and perimeter of the rectangle are $6w$ and $2w + 12$, respectively. Setting these equal gives $w = 3$.

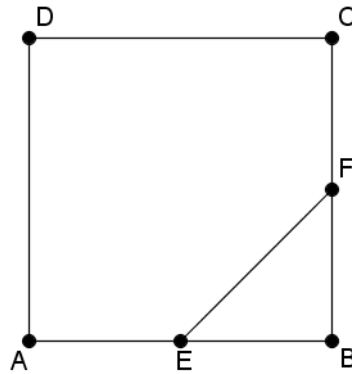
15. (Brian Gu) How many two-digit prime numbers have a ones (units) digit of 9?

Answer (5): First note that to check if a two-digit number is prime, it suffices to check if it is divisible by 2, 3, 5 or 7. This is because a composite number is a product of two numbers both larger than 1 and one of these numbers must be a single digit, hence a multiple of a single digit prime. Among 19, 29, 39, 49, 59, 69, 79, 89, 99, none of them is divisible by 2 or 5. 39, 69, and 99 are divisible by 3. 49 is divisible by 7. Therefore, the remaining 5 numbers 19, 29, 59, 79, and 89 are all primes.

16. (Rahul Sridhar) Alice's rectangle has a perimeter of 14. Bob's rectangle is twice as long and a third as wide and has an area of 8. What is the longer side length of Alice's rectangle?

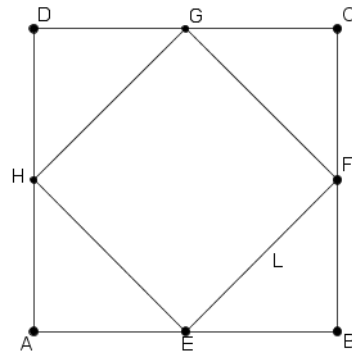
Answer (4): Let a and b be the length and width of Alice's rectangle. Then, Bob's rectangle has length $2a$ and width $b/3$. So, we have $2a + 2b = 14$ and $(2a) \times (b/3) = 8$. We get $a + b = 7$ and $a \times b = 12$. Two numbers with sum 7 and product 12 are 3 and 4. So, Alice's rectangle has dimensions 3 and 4, the longer side being 4.

17. (Saranesh Prembabu) In the diagram below, $ABCD$ is a square with side length 4. E and F are midpoints of AB and BC . Find EF^2 .



Answer (8): Consider the diagram below. The four midpoints of $ABCD$ form the vertices of a smaller square: $EFGH$ with side length L and area L^2 .

Drawing the diagonals of $EFGH$ will divide $ABCD$ into 8 equal triangles. Since $ABCD$ has area $4^2 = 16$, each triangle has area 2. So, $EFGH$, which contains 4 of the triangles, has area 8. So we conclude that L^2 is 8.



18. (Tomas Choi) In a game of Funkyball, a player throws a ball at the board. For every ball thrown, the player can either score 3 points or 5 points. What is the largest integer score that a player cannot attain? For example, a player can attain 13 points if he scores 3 points once and 5 points twice, but a player cannot possibly attain 1 point.

Answer (7): We can check every single score and see if it can be attained. Clearly, 1, 2, and 4 cannot be attained but 3 and 5 can be attained. $6 = 3 + 3$ can be attained, but 7 cannot be attained. $8 = 3 + 5$, $9 = 3 + 3 + 3$, and $10 = 5 + 5$ can be attained. Now, we can attain every score after 10 by scoring 3 points continuously after:

$$8, 11, 14, 17, 20, \dots$$

$$9, 12, 15, 18, 21, \dots$$

$$10, 13, 16, 19, 22, \dots$$

So, 7 is the largest number that cannot be attained as a score.

19. (Stan Zhang) The area of a trapezoid is 20. The vertices of the trapezoid are at $(0, 0)$, $(5, 0)$, $(0, 5)$, and $(x, 5)$, where x is positive number. What is the value of x ?

Answer (3): A trapezoid can be split by drawing a diagonal into two triangles with the same height. Adding the areas of these two triangles, we can see that the area of a trapezoid is the average of the bases times the height. The height of the trapezoid is 5. So, the average of the bases must be $\frac{20}{5} = 4$. The bases have lengths $5 - 0 = 5$ and $x - 0 = x$. Since their average is 4, x must be 3.

20. (Eric Huang) A triangle has side lengths 2, 3, and 4. What is the ratio of the longest height to the shortest height?

Answer (2): Let the area of the triangle be A . Area of a triangle is half of base times height. Since the side lengths are 2, 3, and 4, we find that the altitudes have lengths, $\frac{2A}{2}$, $\frac{2A}{3}$, and $\frac{2A}{4}$. We see that the longest altitude has length A and the shortest has length $\frac{A}{2}$. Therefore the ratio is 2.

21. (Katherine Tian) An arithmetic sequence is a sequence of numbers where the difference between the consecutive terms is constant, such as 5, 7, 9, 11, ... The third term of an arithmetic sequence is 20 and the sum of the first eight terms is 244. What is the first term of the sequence?

Answer (6): The 8 terms form 4 pairs that sum to the same value (i.e. the 1st and 8th terms, the 2nd and 7th terms, the 3rd and 6th terms, and the 4th and 5th terms). This same value must be $244 \div 4 = 61$. Since the 3rd term 20, the 6th term must be $61 - 20 = 41$. From the 3rd term to the 6th term, we jump 3 times which amount to $41 - 20 = 21$. So each jump between two consecutive terms is 7. The 1st term is 2 jumps less than the 3rd term. So it is $20 - (2 \times 7) = 6$.

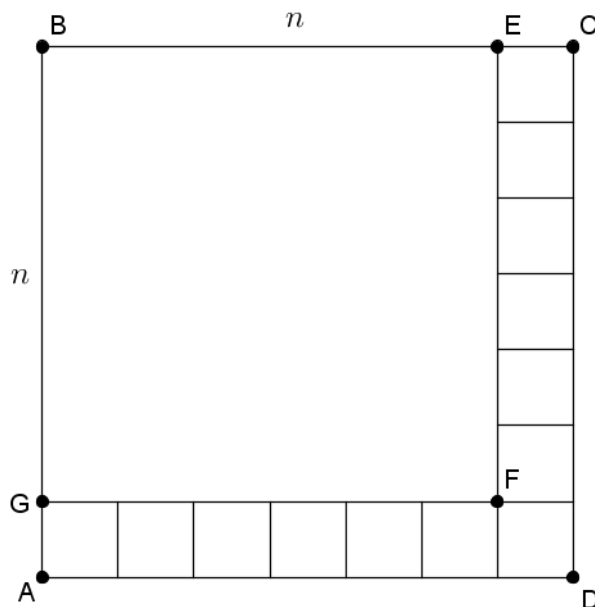
22. (Daniel Yang) The difference of two positive integers is 1. The difference of their squares is 77. What is the ones (units) digit of the larger number?

Answer (9): Let n and $n + 1$ be the consecutive integers. Then

$$(n + 1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1.$$

Thus, $2n + 1 = 77$. So the consecutive integers are $n = 38$ and $n + 1 = 39$. The larger of these is 39 with a ones digit of 9.

For a geometric way to see $(n + 1)^2 - n^2 = 2n + 1$, imagine drawing a square on a grid with side length $n + 1$ and then cutting out a square of side length n . The resulting figure is an L shape with $n + n + 1 = 2n + 1$ unit squares. An example is drawn below for a small value of n .



23. (Stan Zhang) John flips a coin 6 times. What is the difference between the number of ways he can get 4 heads and the number of ways he can get 3 heads? For example, he can get 4 Heads with HHHHTT and 3 Heads with HHHTTT (H represents Heads and T represents Tails).

Answer (5): For the case where he gets 4 heads and 2 tails, there are 6 ways to choose the position of the first tail and 5 ways to choose the position of the second tail. So there seems to be $6 \times 5 = 30$ ways to get 4 Heads and 2 Tails. However, we counted each way to get 2 tails twice. For example HHHHTT is counted once for a tail in 5th position and another in 6th position. It is counted a second time for a tail in 6th position and another in 5th position. Therefore, the number of ways to get 4 heads in a flip of 6 coins is $30 \div 2 = 15$. Similarly, for the case where he gets 3 heads, there are 6 ways to choose the position of the first head, 5 ways to choose the position of the second head, and 4 ways to choose the position of the third head, but we counted each way to get 3 heads $3 \times 2 \times 1$ times, once for each ordering of the 3 heads. So there are $\frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$ ways to get 3 heads. The difference is then $20 - 15 = 5$.

24. (Evan Chen) a, b, c, d are positive integers satisfying

$$(a \times b) + (b \times c) + (c \times d) + (d \times a) = 36.$$

How many possible values of $a + c$ are there?

Answer (7): $ab + bc + cd + da$ factors as $(a + c)(b + d) = 36$. So the possible values are divisors of 36 more than 1 and less than 36. There are 7 of them: 2, 3, 4, 6, 9, 12, and 18.

25. (Stan Zhang) In the equation below, each of the letters H, K, M, O, and T repre-

sents a different digit. H, M, and T cannot be zero. What is the value of M?

$$\begin{array}{rcccc} & M & O & T & H \\ + & T & O & O & K \\ \hline H & M & M & M & M \end{array}$$

Answer (7): First, considering the addition of leading digits, we conclude that H must be 1 because the carry-over when adding numbers is at most 1. Next, the thousands digits addition has digits M and T and the sum has digit M . So T must be 9 (because it cannot be 0). Now, $H + K$ cannot carry over to the tens digit. Otherwise, $T = 9$ would lead to $O = M$. So, the addition of the tens digits without a carry over from the units digits implies that $O = M + 1$. Note that since O is more than M , there is a carry over in both tens digit and hundreds digit additions. Now, the hundreds digit addition gives, $O + O + 1 = 10 + M$. Plugging in $O = M + 1$ reduces this to $2M + 3 = 10 + M$ which results in $M = 7$.

Working out the rest would give $O = 8$ and $K = 6$. So, the original addition in number form is as follows:

$$\begin{array}{rcccc} & 7 & 8 & 9 & 1 \\ + & 9 & 8 & 8 & 6 \\ \hline 1 & 7 & 7 & 7 & 7 \end{array}$$