2015 π Math Contest

TEAM ROUND

SOLUTIONS

1. (Rajiv Movva) Simplify the following expression

$$7 \times \left(\frac{7}{11} + \frac{2}{7} + \frac{4}{11}\right).$$

Answer (9):

$$7 \times \left(\frac{7}{11} + \frac{2}{7} + \frac{4}{11}\right) = 7 \times \left(\frac{7+4}{11} + \frac{2}{7}\right) = 7 \times \left(1 + \frac{2}{7}\right) = 7 + 2 = 9.$$

2. (Mehmet Kaysi) What is $8 \times (0 - 1 + 2 \times 3 \div 4 - 5 + 6 \times 7 \div 8)$?

Answer (6):

$$\begin{aligned} & 8 \times (0 - 1 + 2 \times 3 \div 4 - 5 + 6 \times 7 \div 8) \\ = \; & 8 \times (0 - 1 + 6 \div 4 - 5 + 42 \div 8) \\ = \; & 0 - 8 + 12 - 40 + 42 = 54 - 48 = 6. \end{aligned}$$

3. (Swapnil Garg) An escalator is 30 feet long. If Richard walks up the escalator at the rate of 6 feet per second and the escalator moves down at the rate of 1 foot per second, how many seconds will it take Richard to walk up the escalator?

Answer (6): Because Richard walks up at 6 feet per second, but the escalator travels down at 1 foot per second, each second he goes up $6 - 1 = 5$ feet, so it will take $30/5 = 6$ seconds to reach the top.

4. (Brian Gu) On Wednesday, Sam had some rabbits. On Thursday, he woke up early to find that the number of rabbits had doubled overnight. Thursday afternoon, Sam's friend Brian gave him seven more rabbits. After this, Sam counted all of his rabbits and found that he had 15 rabbits. How many rabbits did Sam have on Wednesday?

Answer (4): If the number of rabbits Sam had on Wednesday was R, then the number of rabbits he had by Thursday morning was $2R$. After Brian gave Sam 7 more rabbits, Sam had $2R + 7$ rabbits. Since we know that, at this point, Sam had 15 rabbits, we can write the equation $2R + 7 = 15$. Solving for R, we find that

$R = 4$.

Alternate Solution. Let us work backwards. On Thursday afternoon, with the 7 rabbits Brian gave him, he had 15 rabbits. So, he had $15 - 7 = 8$ rabbits Thursday morning. Since the number of rabbits doubled Wednesday night, he had $8/2 = 4$ rabbits on Wednesday.

5. (Crystal Su) Aaron the squirrel is drawing a triangle with a base of 15 inches and a height of 4 inches. He accidentally measures the base as 19 inches. By how many square inches did the area change due to this error?

Answer (8): We know that the formula for the area of a triangle is $\frac{b \times h}{2}$ where b is the base length and h is the height. The area of the original triangle is $\frac{15 \times 4}{2} = 30$ square inches. The area of the new triangle is $\frac{19 \times 4}{2} = 38$ square inches. Therefore the difference is $38 - 30 = 8$ square inches.

Alternative Solution. The difference in the original and measured bases is $19 - 15 = 4$ inches. This causes a difference of $\frac{4 \times 4}{2} = 8$ square inches in the area.

6. (Richard Spence) Farmer John has some bicycles and tricycles in his garage. He counts a total of 12 wheels. Given that he owns at least one bicycle and one tricycle, how many bicycles and tricycles does Farmer John have in total?

Answer (5): Note that since the total number of wheels is even and odd number of wheels only come from tricycles, John must have an even number of tricycles. If he has 4 tricycles, then he would have no bicycles. If he has more than 4 tricycles, then he would have more than 12 wheels. So, we conclude that he has 2 tricycles and 3 bicycles, for a total of 5.

7. (Mehmet Kaysi) One angle of a triangle is 20 degrees less than a right angle. Another angle of the same triangle is 15 degrees less than a right angle. Find the measure of the third angle of this triangle, in degrees.

Note: You may use the fact that the sum of the three angles in a triangle is 180 degrees.

Answer (35): The first angle is $90 - 20 = 70$ degrees. The second angle is $90 - 15 = 75$ degrees. Their sum is 145 degrees. Since the three angles add up to 180 degrees, the third one must be 35 degrees.

Alternative Solution. The sum of the first two angles is $(90 - 20) + (90 - 15) = 180 - (20 + 15)$ degrees. Since the sum of all three angles is 180 degrees, the third one must be $20 + 15 = 35$ degrees.

8. (Tomas Choi) Rachel made some cookies for her friends. She gave half her cookies to her best friend Casey. She then gave one third of the remaining cookies to

another friend Bessie. Finally she gave the remaining 6 cookies to Diana. How many cookies did Rachel make?

Answer (18): Work backwards. Before giving her remaining cookies to Diana, Rachel had 6 cookies. So, she had 9 cookies before giving one third (3) to Bessie. Before that, Rachel had 18 cookies and gave half (9) to Casey.

9. (Mayank Pandey) Last year, Mayank's age was a perfect square. Next year, his age will be a perfect cube. How old is Mayank if he is less than 100 years old?

   Note: When a natural number is written *twice* and multiplied, the result is called a *perfect square*. For example $2 \times 2 = 4$ is a perfect square. When a natural number is written *three times* and multiplied, the result is called a *perfect cube*. For example $2 \times 2 \times 2 = 8$ is a perfect cube.

   Answer (26): His current age is one less than a perfect cube and one more than a perfect square. These are the only perfect cubes not more than 100: 1, 8, 27, and 64. Checking each of these cases, it is not hard to see that his current age is one less than 27 and one more than 25. So, Mayank is 26 years old.

10. (Richard Yi) How many two-digit numbers are there whose square is a four-digit number?

    Answer (68): Note that $30^2 = 900$, $31^2 = 961$, and $32^2 = 1024$. So, 32 is the first number whose square is a 4-digit number. Also, $99^2 = 9801$ and $100^2 = 10000$. So, 99 is the last number whose square is a 4-digit number. Among the first 99 positive integers, 31 of them are too small. So, $99 - 31 = 68$ of them work.

11. (Julia Huang) How many two-digit numbers contain even digits only? 42, 66, and 80 are such numbers.

    Answer (20): Let us list all two-digit numbers with even digits:
    20, 22, 24, 26, 28
    40, 42, 44, 46, 48
    60, 62, 64, 66, 68
    80, 82, 84, 86, 88
    So, there are 20 such numbers.

12. (Aaron Lin) Rectangle A is 2015 units by 2015 units. Rectangle B is 2013 units by 2017 units. In square units, what is the positive difference between the area of rectangle A and the area of rectangle B?

    Answer (4): The area of rectangle A is 2015×2015. The area of rectangle B is $2013 \times 2017 = (2015 - 2)(2015 + 2) = 2015 \times 2015 + 2015 \times 2 - 2 \times 2015 - 2 \times 2 = 2015 \times 2015 - 4$. So the difference between these two areas is 4.

    Alternative Solution. Since the numbers are not too big, we can calculate

the value of the areas. The area of rectangle A is $2015 \times 2015 = 4060225$. The area of rectangle B is $2013 \times 2017 = 4060221$. So the difference between these two numbers is 4.

13. (Caleb Ji) How many words, with or without meaning, can we make by rearranging the letters of the "word" DADADA?
ADADAD is an example.

Answer (20): If the first letter is A, we can write the following "words":
AAADDD
AADADD
AADDAD
AADDDA
ADAADD
ADADAD
ADADDA
ADDAAD
ADDADA
ADDDAA
There are 10 "words" that start with A. By symmetry, there are also 10 "words" that start with D. So, the answer is 20.

14. (Rajiv Movva) How many of the first 200 positive integers do not contain the digit 5?

Answer (162): Among the first 100 numbers, 10 of them have 5 in units digit:

$$5, 15, 25, 35, 45, 55, 65, 75, 85, 95$$

and 9 more (excluding 55 which is already counted) have 5 in tens digit:

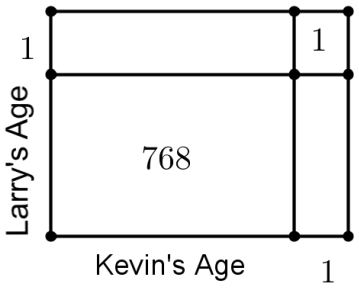
$$50, 51, 52, 53, 54, 56, 57, 58, 59.$$

So, 19 numbers contain the digit 5 among the first 100 numbers. By symmetry, there are also 19 numbers, from 101 to 200, that contain the digit 5. These are basically 100 added to each of the numbers in the previous list. Altogether we have $19 + 19 = 38$ numbers that contain the digit 5 and the remaining $200 - 38 = 162$ numbers do not contain the digit 5.

15. (Kevin Zhang) The product of Kevin's and Larry's ages is 768 now. Next year, the product of their ages will be 825. What is the sum of their ages now?

Answer (56): Consider a rectangle whose length is Kevin's age and width is Larry's age. Its area is given to be 768. Next year, the rectangle will expand to each dimension by 1 unit. We can divide the larger rectangle into 4 rectangles as in the figure below. The areas of the original rectangle and the unit square sum to $768 + 1 = 769$. The area of one of the remaining rectangles is Kevin's age (since its length is Kevin's age and width is 1) and the other has area Larry's age (since its length is 1 and height is Larry's age). So the sum of Kevin's and Larry's ages

is the sum of the areas of the two remaining rectangles, which is $825 - 769 = 56$.



Alternative Solution.
We can factor 768 as

$1\times768 = 2\times384 = 4\times192 = 8\times96 = 16\times48 = 32\times24 = 64\times12 = 128\times6 = 256\times3.$

If we add 1 to both factors and multiply out, the only time we get 825 is with 32×24. Indeed, we have $33 \times 25 = 825$. So, Kevin and Larry are 32 and 24 years old and the sum of their ages is $32 + 24 = 56$.