2015 π Math Contest

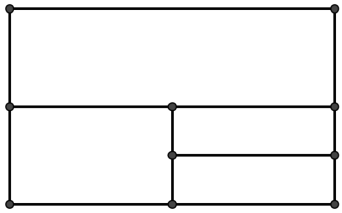
Individual Round

SOLUTIONS

1. (Ali Gurel) $7 - 3 \times 2 =$?

   Answer (1): $7 - 3 \times 2 = 7 - 6 = 1$.

2. (Alicia Weng) In the diagram below, a rectangle is split into two halves by a horizontal line segment. Then the lower rectangle is split into two halves by a vertical line segment. Finally, the lower right rectangle is split into two halves by a horizontal line segment. What is the ratio of the area of the entire rectangle to the area of one of the two smallest rectangles?



   Answer (8): Each rectangle has half the area of the next smallest rectangle. Then the smallest rectangle has area $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ that of the largest one, and so the ratio of the areas of the largest rectangle to the smallest one is 8.

   Alternative Solution. Let the area of the two smallest rectangles be 1 each. Then the area of the lower left rectangle is $1 + 1 = 2$ and the area of the upper rectangle is $2 + 1 + 1 = 4$. The entire rectangle has area $4 + 2 + 1 + 1 = 8$ and the desired ratio is $8/1 = 8$.

3. (Crystal Su) A prime number is a whole number greater than 1 whose only two whole-number factors are 1 and itself. How many prime numbers are divisible by either 2, 3, or 4?

   Answer (2): Primes, by definition, are only divisible by 1 and themselves. Therefore, the only primes divisible by 2 and 3 would be 2 and 3, respectively. No primes are divisible by 4, because 4 itself is a composite number. So, there are only 2 such primes.

4. (Tomas Choi) Kevin has a 65% off coupon which he wants to use to purchase new shoes. If the shoes originally cost \$20, how many dollars does he need to spend on the shoes using the coupon?

   Answer (7): The coupon saves 65% so he needs to pay 35% of \$20 which is 7 dollars.

5. (Ali Ersoz) 1% of a number is what percent of 25% of that number?

   Answer (4): 1% is 1/25 of 25% and 1/25 is the same as 4%. So, the answer is 4.

6. (Andrew Lin) The sum of Andrew's and Phoebe's ages is 16. Andrew was 2 years old when Phoebe was born. How old is Andrew now?

   Answer (9): Their current ages are two numbers which sum to 16 and differ by 2. So, they are 7 and 9 and Andrew is 9 years old.

7. (Kevin Zhang) You can rearrange the letters in the "word" *bee* in three different ways: *bee*, *ebe*, and *eeb*. In how many ways can you rearrange the letters in the "word" *red*?

   Answer (6): There are 6 of them: *red*, *rde*, *der*, *dre*, *erd*, and *edr*.

8. (Benjamin Chen) Waves crash on a beach at a rate of 1 wave every 35 seconds. If Aaron the squirrel sits on the beach for 5 minutes, how many waves will he see crashing onto the beach if Aaron sees his first wave at the end of the first 5 seconds?

   Answer (9): Aaron is on the beach for $5 \times 60 = 300$ seconds. He sees a wave at 5 seconds, $5+35 = 40$ seconds, $5+2\times35 = 75$ seconds and so on until $5+8\times35 = 285$ seconds, for a total of 9 waves.

9. (Ali Gurel) In a group of 15 students, 5 like to play football, 6 like to play basketball and 7 like to play neither. How many students like to play both football and basketball?

   Answer (3): Leaving those 7 students aside, we have 8 students left who like to play at least one sport. Three of these students do not like football, so they like only basketball. Among the 6 basketball fans, the remaining $6 - 3 = 3$ students must like both basketball and football.

10. (Crystal Su) Bob is mowing lawns over the summer. If he can mow a 10-foot by 10-foot lawn in 40 minutes, how many hours will it take him to mow a 12-foot by 25-foot lawn, given that he always mows at the same constant rate?

    Answer (2): We know that Bob mows a 10-foot by 10-foot or a 100-square foot

lawn in 40 minutes. A 12-foot by 25-foot lawn has an area of 300 square feet. Since $300 = 3 \times 100$, it takes him $3 \times 40 = 120$ minutes or 2 hours to move 300 square feet.

11. (Evan Chen) What is the sum of the digits of $1001 \times 1001 - 1000 \times 1000$?

Answer (3): By direct computation, we get

$$1001 \times 1001 - 1000 \times 1000 = 1,002,001 - 1,000,000 = 2001.$$

So, the answer is $2 + 0 + 0 + 1 = 3$.

Alternative Solution.

$$\begin{aligned} & 1001 \times 1001 - 1000 \times 1000 = (1000 + 1) \times (1000 + 1) - 1000 \times 1000 \\ = \; & 1000 \times 1000 + 1000 + 1000 + 1 - 1000 \times 1000 \\ = \; & 1000 + 1000 + 1 = 2001. \end{aligned}$$

So, the answer is 3.

12. (Brian Gu) For how many minutes from midnight (12:00 AM) to noon (12:00 PM) on a standard 12-hour digital clock is the value of the hour equal to exactly twice the value of the minute? One example is the minute 4:02, since the hour (4) is twice the minute (2).

Answer (6): For the hour to be exactly twice the minute, the hour must be an even number. Thus, the only possible values for the hour are 12, 2, 4, 6, 8, 10, in order of appearance. For each of these hours, there is exactly one minute during the hour where the problem condition is satisfied: 12:06, 2:01, 4:02, 6:03, 8:04, and 10:05. Therefore, there are exactly 6 minutes where the value of the hour is equal to exactly twice the value of the minute.

13. (Crystal Su) Adam is 13 years younger than Laura. Laura is twice as old as Jessica. Jessica is 5 years older than Kelvin. If Kelvin is 2 years old, how old is Adam?

Answer (1): Let us work backwards. Kelvin is 2 years old and Jessica is 5 years older. So, Jessica is 7 years old. Laura is twice as old as Jessica. So, Laura is 14 years old. Finally, Adam is 13 years younger than the 14 year old Jessica. So, Adam must be 1 year old.

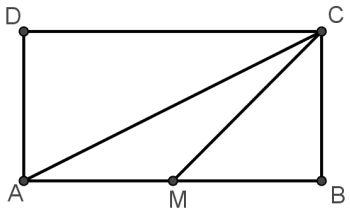
14. (Benjamin Chen) Mario is collecting mushrooms. There are 3 kinds of mushrooms and he has to pay for them with coins. A red mushroom costs 5 coins, a blue mushroom costs 4 coins, and a yellow mushroom costs 1 coin. What is the smallest number of mushrooms he can collect with exactly 23 coins?

Answer (5): Note that with 4 mushrooms, he can get at most $4 \times 5 = 20$ coins. So, he needs at least 5 mushrooms. 3 red and 2 blue mushrooms have a total value of $3 \times 5 + 2 \times 4 = 23$ coins. So, the answer is 5.

15. (Mehmet Kaysi) The sum of the factors of a positive integer is calculated by adding all the factors of the number. For example, the sum of the factors of 10 is $1+2+5+10=18$. For how many positive 1-digit numbers, $\{1,2,3,4,5,6,7,8,9\}$, is the sum of the factors one more than the digit itself?

    Answer (4): A digit that satisfies the condition should not have any factors besides 1 and itself. So, it has to be a prime number. There are 4 prime digits: 2, 3, 5, and 7.

16. (Evan Chen) In the diagram below, $ABCD$ is a rectangle and M is the midpoint of AB. If triangle AMC has area 2, what is the area of rectangle $ABCD$?

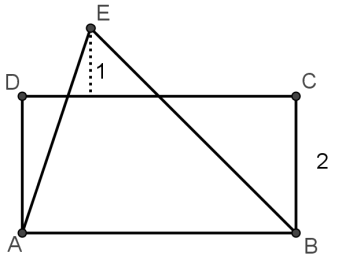


    Answer (8): Note that $[AMC] = [MBC] = 2$. So, $[ABC] = 2 + 2 = 4$ and $[ABCD] = 4 + 4 = 8$.

17. (Benjamin Chen) Allen flips a coin 9 times. If it lands heads, he gets 3 points. If it lands tails, he gets 1 point. After 9 flips, Allen has 13 points. How many times did the coin land tails?

    Answer (7): If Allen have tails in all 9 flips, he would have 9 points. Each time we replace a tails by a heads, he receives 2 additional points. Since he needs 4 more points to get to 13 points, we need to replace 2 of the tails by heads. So, he flipped 2 heads and 7 tails.

18. (Richard Spence) In the image below, $ABCD$ is a rectangle with $BC = 2$ units. Point E is above segment CD such that the distance from E to CD is 1 unit. If the area of rectangle $ABCD$ is 8 square units, how many square units is the area of triangle ABE?



Answer (6): $ABCD$ has area 8 and width $BC = 2$ so it has length $AB = 4$. The base of $\triangle ABE$ is $AB = 4$ and its height is the distance from E to AB which is $2 + 1 = 3$. The area of a triangle is half the product of its base and height. So, $[ABE] = \frac{1}{2} \times 4 \times 3 = 6$.

19. (Mehmet Kaysi) In how many ways can 9 be written as a sum of two or more prime numbers if the order of the primes in the sum does not matter? A prime number can be used more than once. For example, $9 = 2 + 2 + 5$ is one way to write it.

Answer (4): There are 4 ways:

$$7 + 2,\ 5 + 2 + 2,\ 3 + 3 + 3,\ 3 + 2 + 2 + 2.$$

20. (Benjamin Chen) A squirrel family has less than 60 members. When the squirrels are arranged in 3 rows, 2 of them are left out. When they are arranged in 4 rows, 3 of them are left out. When they are arranged in 5 rows, none of them is left out. How many squirrels are in each row when they are arranged in 5 rows?

Answer (7): We can check multiples of 5. 5 does not work, and neither do 10, 15, 20, 25, and 30. 35 does work; it leaves a remainder of 2 when divided by 3 and a remainder of 3 when divided by 4. So, the answer is $35/5 = 7$. One can further check that none of the remaining multiples of 5: 40, 45, 50, 55 work.

21. (Tiancheng Qin) What is the ones digit of the sum of the first 30 positive integers?

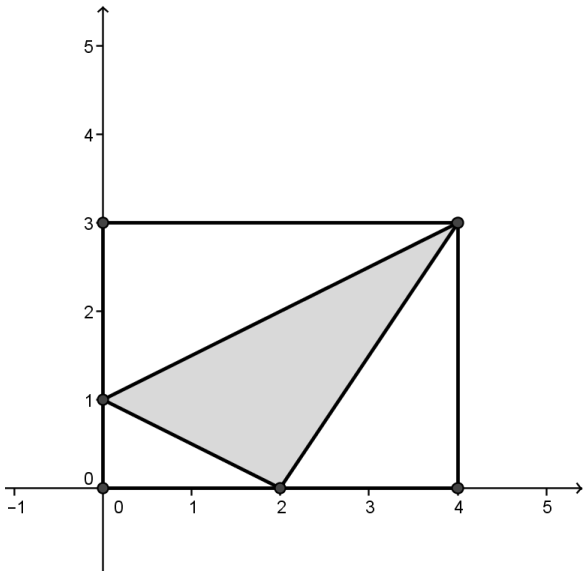
Answer (5): If we pair the numbers as (1,29), (2, 28), (3, 27), etc. the only numbers left are 15 and 30. The two numbers in each pair sum to 30 and thus contributes nothing to the ones digit. The remaining two numbers 15 and 30 add up to 45. Hence, the ones digit of the sum is 5.

22. (Andrew Lin) In heptagon $SWAPNIL$ with 7 sides, $\angle S = \angle W = \angle A = \angle P = \angle N = \angle I = 149$ degrees. How many degrees is $\angle L$?

Note: You may use the fact that the sum of the three angles in a triangle is 180 degrees.

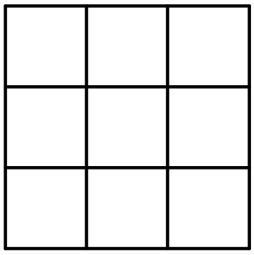
Answer (6): By drawing diagonals from one vertex, we can split the heptagon into 5 triangles. Each triangle has sum of angles 180 degrees. So, the sum of the angles of the heptagon is $180 \times 5 = 900$ degrees. Excluding $\angle L$, the remaining angles add up to $(150 - 1) \times 6 = 900 - 6$ degrees. So, $\angle L = 6$ degrees.

23. (Julia Huang) What is the area of the shaded triangle below?



Answer (4): We can find the area of the triangle by finding the area of the rectangle and subtracting the missing triangles.
The area of the rectangle is $3 \times 4 = 12$.
The area of the lower left triangle is $\frac{1 \times 2}{2} = 1$.
The area of the triangle on the right is $\frac{2 \times 3}{2} = 3$.
The area of the upper triangle is $\frac{2 \times 4}{2} = 4$.
So the area of the shaded triangle is $12 - 1 - 3 - 4 = 12 - 8 = 4$.

24. (Rahul Sridhar) What is the difference between the number of squares and the number of rectangles that are not squares in the diagram below?



Answer (8): There are 9 unit squares, four 2×2 squares and one 3×3 square. So, number of squares is $9 + 4 + 1 = 14$. There are six 2×1 rectangles, three 3×1 rectangles and two 3×2 rectangles. So, there are $6 + 3 + 2 = 11$ rectangles with length larger than width. Similarly, there are 11 rectangles with length smaller than width. So, there are 22 non-square rectangles. The desired difference is $22 - 14 = 8$.

25. (Alicia Weng) Archibald and Cornelius each has a total of twelve pets, with each pet being either an octopus with eight legs or a turtle with four legs. Cornelius has twice as many octopi as Archibald does, and the total number of legs his pets have is twenty more than the total number of legs Archibald's pets have. How many octopi does Archibald have?

Answer (5): An octopus has 4 more legs than a turtle. Since the number of Cornalius's pet have 20 more legs, he must have 5 more octopi than Archibald. This is only possible when Archibald has 5 octopi, and Cornelius have 10 octopi, since Cornelius has twice as many octopi as Archibald does.