Topics & Sample Problems

MC35F (AMC 10/12 Advanced Fundamentals)

alphastar academy

Part-I

MC35F-1 Algebra

Each chapter is followed by a short summary of topics covered in that chapter, along with one sample contest problem from that chapter. Supplemental topics introduced in MC35F are labeled with an asterisk (*).

Chapter 1: Arithmetic

- Word problems involving integers, fractions, decimals, and percentages
- Terminating and non-terminating decimals
- Real, rational, and irrational numbers, irrationality of $\sqrt{2}$

(AMC10-2022-A17) How many three-digit positive integers $\underline{a}\ \underline{b}\ \underline{c}$ are there whose nonzero digits a, b, and c satisfy

$$0.\overline{\underline{a}\ \underline{b}\ \underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$$

(The bar indicates repetition, thus $0.\overline{\underline{a}\ \underline{b}\ \underline{c}}$ is the infinite repeating decimal $0.\underline{a}\ \underline{b}\ \underline{c}\ \underline{a}\ \underline{b}\ \underline{c}\cdots$)

Chapter 2: Exponents & Radicals

- Defn., properties of exponents, radicals
- Negative, fractional, real exponents
- AM-GM inequality (*)

(Lehigh MC-2016-34) What is the smallest integer larger than $(\sqrt{5}+\sqrt{3})^6$? Note that $3.872 < \sqrt{15} < 3.873$.

Chapter 3: Word Problems & System of Equations

- Logic (converse, contrapositive, etc.)
- Systems of equations

(AMC10-2002-B20) Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

Chapter 4: Distance, Rate, and Time

- $d = rt$, average speed, relative speed
- Harmonic mean
- Work/output problems
- AM-HM inequality (*)

(AIME-2024-I-1) Every morning, Aya does a 9 kilometer walk, and then finishes at the coffee shop. One day, she walks at s kilometers per hour, and the walk takes 4 hours, including t minutes at the coffee shop. Another morning, she walks at $s + 2$ kilometers per hour, and the walk takes 2 hours and 24 minutes, including t minutes at the coffee shop. This morning, if she walks at $s + \frac{1}{2}$ kilometers per hour, how many minutes will the walk take, including the t minutes at the coffee shop?

Chapter 5: Sequences-1

- Review of statistical definitions (mean, median, mode, range)
- Arithmetic and geometric sequences, series
- Arithmetico-geometric series (*)

(Lehigh MC-2004-36) What is the sum of the reciprocals of all positive integers none of whose prime divisors is greater than 3? That is, what is the value of

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \cdots ?$$

Chapter 6: Sequences-2

- Recursive sequences, using patterns to find the n^{th} term of a sequence
- Fibonacci sequence
- Using induction to prove formulas for the n^{th} term of a recursive sequence (*)

(AMC10-2024-B23) The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}?$$

Chapter 7: Functions & Operations

- Function definitions (domain, codomain, range, etc.)
- Operations
- Functional equations

(AMC12-2023-B22) A real-valued function f has the property that for all real numbers a and b,

$$f(a+b) + f(a-b) = 2f(a)f(b).$$

Which one of the following cannot be the value of $f(1)$?

Chapter 8: Polynomials-1

- Univariate polynomials (e.g., $x^2 - 4x + 1$)
- Solving quadratic equations by factoring, completing the square, or the Quadratic Formula
- Defn. of parabola, focus, directrix
- Discriminant, vertex of a parabola
- Fundamental Theorem of Algebra, Rational Root Theorem
- Finite differences method (*)

(AMC12-2010-A21) The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of x, where the graph and the line intersect. What is the largest of these values?

Chapter 9: Polynomials-2

- Factor Theorem, Remainder Theorem
- Vieta's Formulas for quadratic, cubic, and degree-d polynomials

(AMC12-2024-A15) The roots of $x^3 + 2x^2 - x + 3$ are p, q, and r. What is the value of

$$(p^2+4)(q^2+4)(r^2+4)?$$

Chapter 10: Trigonometry

- Radians, right triangle and unit circle definitions of sin, cos, tan, etc.
- Trigonometric identities (addition/subtraction formulas, double-angle, power-reduction, etc.)
- Trigonometric substitution
- Advanced identities, e.g., $\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \ldots + \cos\frac{(n-1)\pi}{n} = -\frac{1}{2}$ (*)

(AMC12-2022-A17) Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a that can be written in the form $(p, q) \cup (q, r)$, where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?

Chapter 11: Logarithm

- Definition of $\log_b a$, logarithmic identities (change-of-base, addition and subtraction of logarithms, etc.
- Natural logarithms (ln), the number e

(AIME-2024-I-2) There exist real numbers x and y, both greater than 1, such that $\log_x\left(y^x\right) = \log_y\left(x^{4y}\right) = 10$. Find xy.

Chapter 12: Complex Numbers

- Rectangular form ($a + bi$), complex conjugate, magnitude, properties of complex numbers
- Conjugate Root Theorem
- Polar form, Euler's and de Moivre's formulas, roots of unity

(AMC12-2024-B12) Suppose z is a complex number with positive imaginary part, with real part greater than 1, and with $|z| = 2$. In the complex plane, the four points 0, z, z^2, and z^3 are the vertices of a quadrilateral with area 15. What is the imaginary part of z?

MC35F-1 Counting

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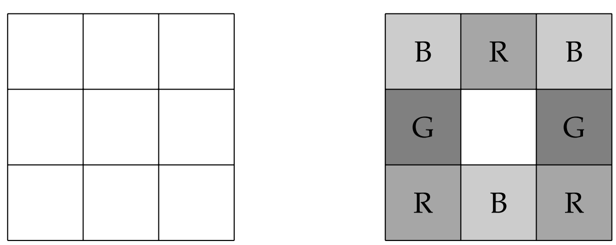
Chapter 1: Counting Basics

- Addition, Multiplication Principles
- Permutations, factorials ($n!$)
- Permutations k at a time $P(n, k)$, combinations $\binom{n}{k}$

(AMC10-2023-A20) Each square in a 3×3 grid of squares is colored red, white, blue, or green so that every 2×2 square contains one square of each color. One such coloring is shown on the right below. How many different colorings are possible?



Chapter 2: Casework

- Solving difficult counting (enumeration) problems by breaking them into disjoint cases

(AMC10-2024-B20) Three different pairs of shoes are placed in a row so that no left shoe is next to a right shoe from a different pair. In how many ways can these six shoes be lined up?

Chapter 3: Complementary Counting & Overcounting

- Complementary counting, overcounting (e.g., number of distinct seatings around a table)
- Multinomial coefficients $\binom{n}{a_1, a_2, \ldots, a_k}$

(AIME-2017-I-1) Fifteen distinct points are designated on $\triangle ABC$: the 3 vertices A, B, and C; 3 other points on side $\overline{AB}$; 4 other points on side $\overline{BC}$; and 5 other points on side $\overline{CA}$. Find the number of triangles with positive area whose vertices are among these 15 points.

Chapter 4: Counting Sets

- Set definitions and notation (e.g., $\in$, $\subseteq$, $\subset$, $\cup$, $\cap$, $|A|$)
- Subsets, number of subsets of a set
- Principle of Inclusion-Exclusion for 2 or 3 sets
- Derangements, formula for $!n$ (*)

(HMMT Nov-2014-Guts-20) Determine the number of sequences of sets $S_1, S_2, \ldots, S_{999}$ such that

$$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_{999} \subseteq \{1, 2, \ldots, 999\}.$$

Here, $A \subseteq B$ means that all elements of A are also elements of B.

Chapter 5: Counting with Digits

- Solving problems involving finding the number of n-digit integers satisfying some property
- Palindromes, numbers with increasing or decreasing digits

(AMC10-2017-A25) How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property.

Chapter 6: Path Counting & Bijections

- Number of paths in a lattice grid
- Definition of bijection (one-to-one correspondence)
- Solving counting problems by finding a bijection

(AMC12-2010-A18) A 16-step path is to go from $(-4,-4)$ to $(4,4)$ with each step increasing either the x-coordinate or y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2, -2 \le y \le 2$ at each step?

Chapter 7: Stars and Bars

- Stars and Bars derivation, formulas (both positive and non-negative variants)

(AMC10-2016-A20) For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?

Chapter 8: Binomial

- Expansion of $(x+y)^n$, Binomial Theorem, Pascal's triangle
- Combinatorial identities (e.g., Pascal's Identity, Hockey Stick Identity), double counting
- Multinomial Theorem
- Double counting (*)

(AMC10-2011-B23) What is the hundreds digit of 2011^{2011}?

Chapter 9: Counting with Recursion

- Defn. of recursive sequence, Fibonacci sequence
- Solving counting problems recursively
- Catalan numbers C_n

(AMC12-2015-A22) For each positive integer n, let $S(n)$ be the number of sequences of length n consisting solely of the letters A and B, with no more than three As in a row and no more than three Bs in a row. What is the remainder when $S(2015)$ is divided by 12?

Chapter 10: Probability-1

- Probability axioms, definitions (e.g., independent events, disjoint events)
- Binomial distribution ($P(k$ out of n successes$)$)
- Recursive probability

(AMC10-2023-B21) Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

Chapter 11: Probability-2

- Conditional probability, Bayes' Theorem
- Geometric probability in 2 and 3 dimensions

(AMC10-2018-B22) Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. Which of the following numbers is closest to the probability that x, y, and 1 are the side lengths of an obtuse triangle?

Chapter 12: Expected Value

- Random variables, defn. of expected value (finitely or countably infinitely many outcomes)
- Linearity of expectation
- Expected number of trials before 1^{st} success

(AMC10-Fall-2021-B16) Five balls are arranged around a circle. Chris chooses two adjacent balls at random and interchanges them. Then Silva does the same, with her choice of adjacent balls to interchange being independent of Chris's. What is the expected number of balls that occupy their original positions after these two successive transpositions?