# Topics & Sample Problems

## MC30F (AMC 10/12 Basic Fundamentals)

alphastar academy

# Part-I

# MC30F-1 Algebra

Each chapter is followed by a short summary of topics covered in that chapter, along with one sample contest problem from that chapter.

## Chapter 1: Arithmetic

- Word problems involving integers, fractions, decimals, and percentages
- Terminating and non-terminating decimals
- Real, rational, and irrational numbers, irrationality of $\sqrt{2}$

(AMC12-2013-A10) Let $S$ be the set of positive integers $n$ for which $\frac{1}{n}$ has the repeating decimal representation $0.\overline{ab} = 0.ababab\cdots$, with $a$ and $b$ different digits. What is the sum of the elements of $S$?

## Chapter 2: Exponents & Radicals

- Defn., properties of exponents, radicals
- Negative, fractional, real exponents

(AMC10-2022-A11) Ted mistakenly wrote $2^m \cdot \sqrt{\frac{1}{4096}}$ as $2 \cdot \sqrt[m]{\frac{1}{4096}}$. What is the sum of all real numbers $m$ for which these two expressions have the same value?

## Chapter 3: Word Problems

- Logic (converse, contrapositive, etc.)
- Systems of equations

(SMT-2018-Algebra-1) At the grocery store, 3 avocados and 2 pineapples cost \$8.80, while 5 avocados and 3 pineapples cost \$14.00. In dollars and cents, how much does one avocado and one pineapple cost?

## Chapter 4: Distance, Rate, and Time

- $d = rt$, average speed, relative speed
- Harmonic mean
- Work/output problems

(AMC10-2010-B10) Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

## Chapter 5: Sequences-1

- Review of statistical definitions (mean, median, mode, range)
- Arithmetic and geometric sequences, series

(AMC12-2009-A17) Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is $r_1$, and the sum of the second series is $r_2$. What is $r_1 + r_2$?

## Chapter 6: Sequences-2

- Recursive sequences, using patterns to find the $n^{\text{th}}$ term of a sequence
- Fibonacci sequence

(Math Day at the Beach-2018-Individual-14) Form the sequence such that $x_1 = x_2 = 1$, and for $n > 2$, $x_n = x_{n-1}^2 + x_{n-2}$. Of the numbers $x_1,\ x_2, \ldots, x_{2018}$, how many are divisible by 3?

## Chapter 7: Functions & Operations

- Function definitions (domain, codomain, range, etc.)
- Operations
- Functional equations

(AMC12-2019-B8) Let $f(x) = x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - f\left(\frac{4}{2019}\right) + \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)?$$

## Chapter 8: Polynomials-1

- Univariate polynomials (e.g., $x^2 - 4x + 1$)
- Solving quadratic equations by factoring, completing the square, or the Quadratic Formula
- Defn. of parabola, focus, directrix
- Discriminant, vertex of a parabola
- Fundamental Theorem of Algebra, Rational Root Theorem

(AMC12-2015-A18) The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of $a$?

## Chapter 9: Polynomials-2

- Factor Theorem, Remainder Theorem
- Vieta's Formulas for quadratic, cubic, and degree-$d$ polynomials

(AMC12-2023-B14) For how many ordered pairs $(a, b)$ of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

## Chapter 10: Trigonometry

- Radians, right triangle and unit circle definitions of sin, cos, tan, etc.
- Trigonometric identities (addition/subtraction formulas, double-angle, power-reduction, etc.)
- Trigonometric substitution

(AMC12-Spring-2021-B13) How many values of $\theta$ in the interval $0 < \theta \leq 2\pi$ satisfy

$$1 - 3\sin\theta + 5\cos 3\theta = 0?$$

## Chapter 11: Logarithms

- Definition of $\log_b a$, logarithmic identities (change-of-base, addition and subtraction of logarithms, etc.
- Natural logarithms (ln), the number $e$

(AMC12-2020-A10) There is a unique positive integer $n$ such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of $n$?

## Chapter 12: Complex Numbers

- Rectangular form ($a + bi$), complex conjugate, magnitude, properties of complex numbers
- Conjugate Root Theorem
- Polar form, Euler's and de Moivre's formulas, roots of unity

(AMC12-2022-B11) Let $f(n) = \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$, where $i = \sqrt{-1}$. What is $f(2022)$?

# MC30F-1 Counting

Each chapter is followed by a short summary of topics covered in that chapter, along with one sample contest problem from that chapter.

## Chapter 1: Counting Basics

- Addition, Multiplication Principles
- Permutations, factorials ($n!$)
- Permutations $k$ at a time $P(n,k)$, combinations $\binom{n}{k}$

(AMC10-2013-A11) A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

## Chapter 2: Casework

- Solving difficult counting (enumeration) problems by breaking them into disjoint cases

(AMC10-2020-B17) There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

## Chapter 3: Complementary Counting & Overcounting

- Complementary counting, overcounting (e.g., number of distinct seatings around a table)
- Multinomial coefficients $\binom{n}{a_1, a_2, \ldots, a_k}$

(AMC12-2012-B12) How many sequences of zeros and/or ones of length 20 have all zeros consecutive, or all the ones consecutive, or both?

## Chapter 4: Counting Sets

- Set definitions and notation (e.g., $\in$, $\subseteq$, $\subset$, $\cup$, $\cap$, $|A|$)
- Subsets, number of subsets of a set
- Principle of Inclusion-Exclusion for 2 or 3 sets

(AMC12-2023-B8) How many nonempty subsets $B$ of $\{0, 1, 2, 3, \ldots, 12\}$ have the property that the number of elements in $B$ is equal to the least element of $B$? For example, $B = \{4, 6, 8, 11\}$ satisfies the condition.

## Chapter 5: Counting with Digits

- Solving problems involving finding the number of $n$-digit integers satisfying some property
- Palindromes, numbers with increasing or decreasing digits

(AMC10-2023-B16) Define an *upno* to be a positive integer of 2 or more digits where the digits are strictly increasing moving from left to right. Similarly, define a *downno* to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let $U$ equal the total number of upnos and $D$ equal the total number of downnos. What is $|U - D|$?

## Chapter 6: Path Counting & Bijections

- Number of paths in a lattice grid
- Definition of bijection (one-to-one correspondence)
- Solving counting problems by finding a bijection

(AMC10-2019-A17) A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

## Chapter 7: Stars and Bars

- Stars and Bars derivation, formulas (both positive and non-negative variants)

(AMC10-2018-A11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where $n$ is a positive integer. What is $n$?

## Chapter 8: Binomial

- Expansion of $(x+y)^n$, Binomial Theorem, Pascal's triangle
- Combinatorial identities (e.g., Pascal's Identity, Hockey Stick Identity), double counting
- Multinomial Theorem

(HMMT Feb-2004-Guts-1) Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

## Chapter 9: Counting with Recursion

- Defn. of recursive sequence, Fibonacci sequence
- Solving counting problems recursively
- Catalan numbers $C_n$

(AMC12-2009-B21) Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

## Chapter 10: Probability-1

- Probability axioms, definitions (e.g., independent events, disjoint events)
- Binomial distribution ($P(k$ out of $n$ successes$)$)
- Recursive probability

(AMC12-2015-B17) An unfair coin lands on heads with a probability of $\frac{1}{4}$. When tossed $n > 1$ times, the probability of exactly two heads is the same as the probability of exactly three heads. What is the value of $n$?

## Chapter 11: Probability-2

- Conditional probability, Bayes' Theorem
- Geometric probability in 2 and 3 dimensions

(AMC10-2020-A16) A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0)$, $(2020,0)$, $(2020,2020)$, and $(0,2020)$. The probability that the point is within $d$ units of a lattice point is $\frac{1}{2}$. (A point $(x,y)$ is a lattice point if $x$ and $y$ are both integers.) What is $d$ to the nearest tenth?

## Chapter 12: Expected Value

- Random variables, defn. of expected value (finitely or countably infinitely many outcomes)
- Linearity of expectation
- Expected number of trials before $1^{\text{st}}$ success

(AMC12-Fall-2021-B23) What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1,2,3,\ldots,30\}$? (For example the set $\{1,17,18,19,30\}$ has 2 pairs of consecutive integers.)