Topics & Sample Problems

MC50F (USA(J)MO Fundamentals)

alphastar academy

Contents

Part-II

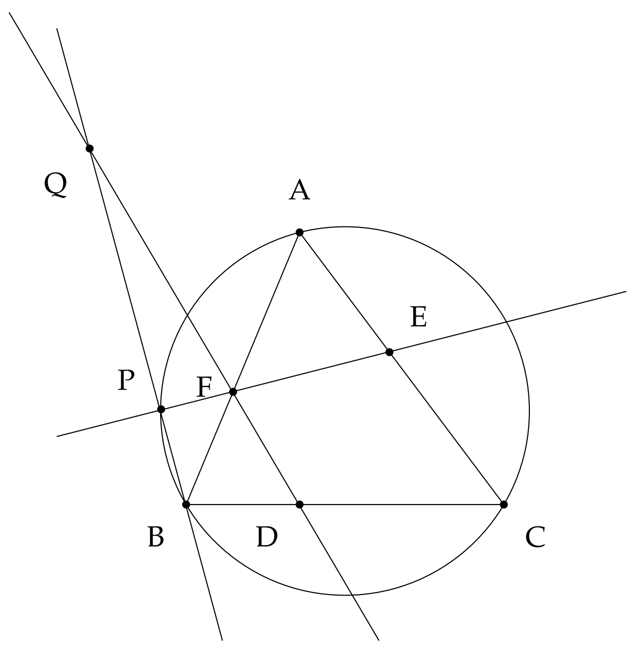
MC50F Geometry

Chapter 1: Angle Chasing

- Developing basic angle chasing techniques
- Incenters, orthocenters, cyclic quadrilaterals
- Directed angles

Sample Problem:

(ISL-2010-G1) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

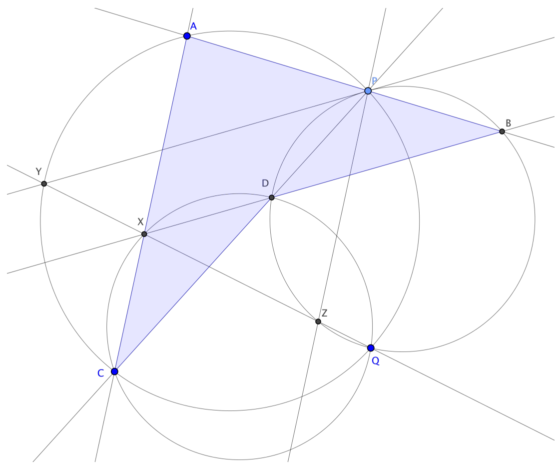


Chapter 2: Similar Triangles

- Using spiral similarity and regular similar triangles to solve geometry problems

Sample Problem:

(USA-TST-2007-1) Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.

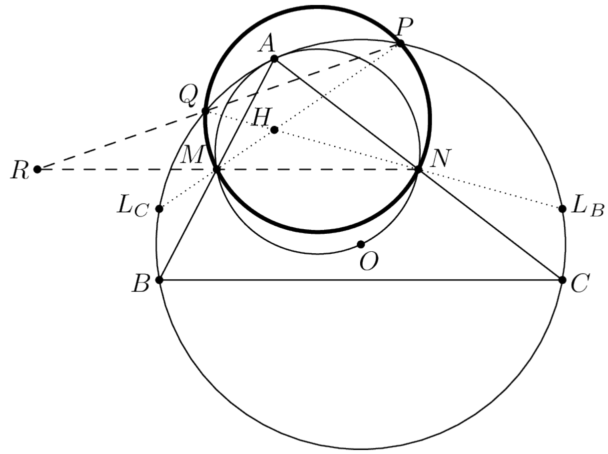


Chapter 3: Power of a Point

- Defining positive/negative similarity and directed lengths
- Defining power, radical axes, and radical centers

Sample Problem:

(TSTST-2011-4) Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.



Chapter 4: Homothety

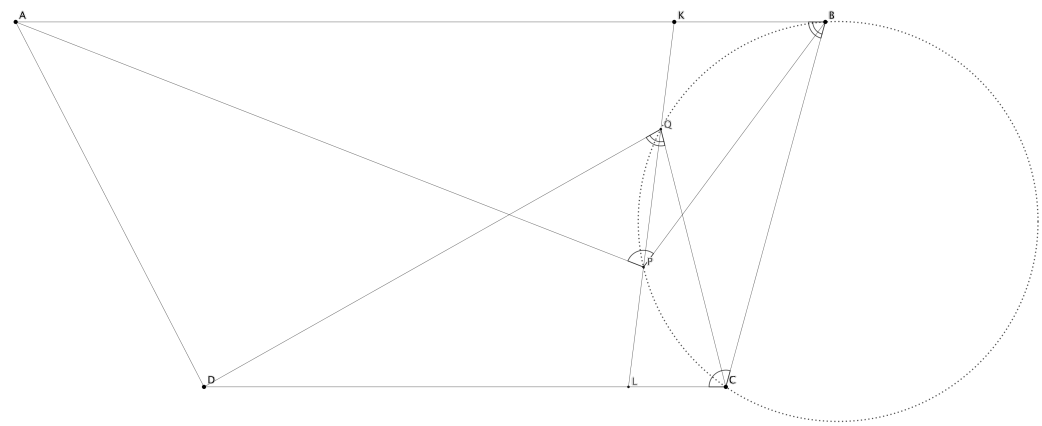
- Using homothety to solve olympiad geometry problems

Sample Problem:

(ISL-2006-G2) Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

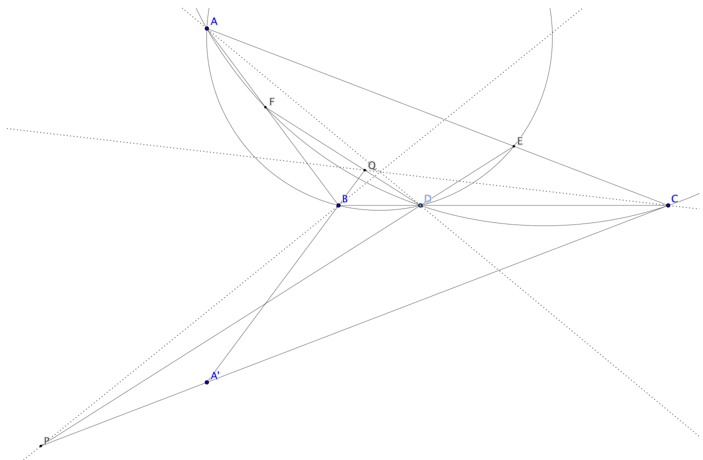


Chapter 5: Collinearity

- Proving collinearity with Menelaus, Pascal, and Desargues' theorems

Sample Problem:

(RMM-2016-1) Let ABC be a triangle and let D be a point on the segment $BC, D \neq B$ and $D \neq C$. The circle ABD meets the segment AC again at an interior point E. The circle ACD meets the segment AB again at an interior point F. Let A' be the reflection of A in the line BC. The lines $A'C$ and DE meet at P, and the lines $A'B$ and DF meet at Q. Prove that the lines AD, BP and CQ are concurrent (or all parallel).



Chapter 6: Concurrency

- Proving concurrency using Ceva and Brianchon's theorems
- Isogonal and isotomic conjugates

Sample Problem:

(ISL-2000-G3) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

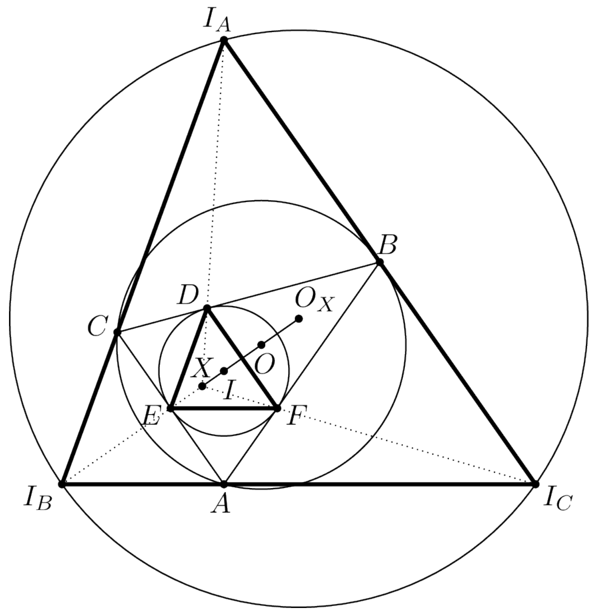
and the lines AD, BE, and CF are concurrent.

Chapter 7: Circles of the Triangle

- Nine-point circle, incircle, excircles and their properties

Sample Problem:

(Vietnam-TST-2003-5) Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

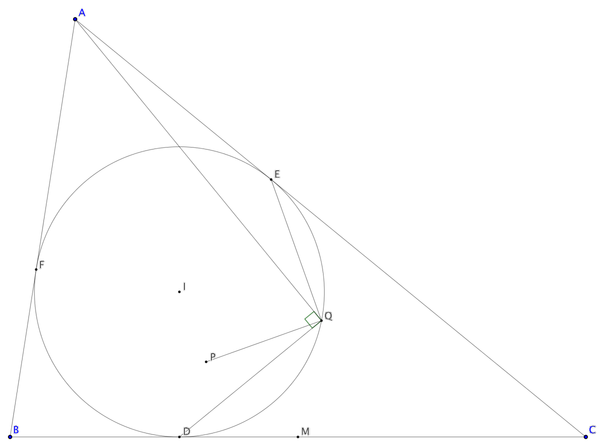


Chapter 8: Configurations-I

- Examining and proving facts about common olympiad geometry configurations

Sample Problem:

(USA-TST-2015-1) Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90°$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90°$ or $\angle PQF = 90°$.

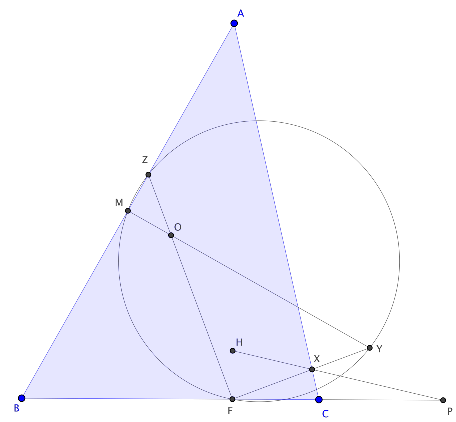


Chapter 9: Configurations-II

- Examining and proving facts about other common olympiad geometry configurations

Sample Problem:

(BMO-2008) Given a scalene acute triangle ABC with $AB > AC$ let F be the foot of the altitude from A. Let P be a point on BC different from B so that $BF = PF$. Let H, O, M be the orthocenter, circumcenter of $\triangle ABC$ and the midpoint of AB respectively. Let X be the intersection of CA and HP, and let Y be the intersection of OM and FX and let OF intersect AB at Z. Prove that F, M, Y, Z are concyclic.



Chapter 10: Configurations-III

- Examining and proving facts about symmedians and mixtilinear incircles

Sample Problem:

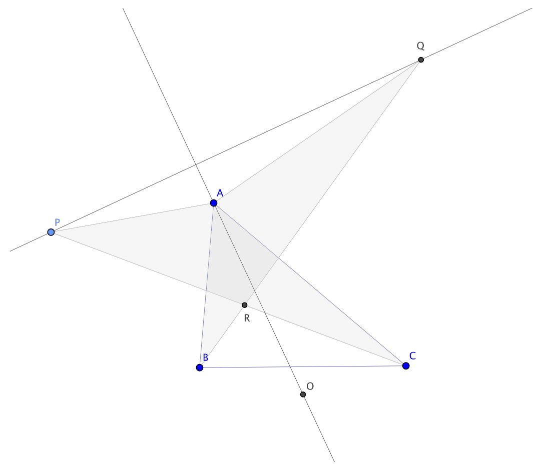
(EGMO-2013-5) Let Ω be the circumcircle of the triangle ABC. The circle ω is tangent to the sides AC and BC, and it is internally tangent to the circle Ω at the point P. A line parallel to AB intersecting the interior of triangle ABC is tangent to ω at Q.
Prove that $\angle ACP = \angle QCB$.

Chapter 11: Complex Numbers

- Developing techniques to solve olympiad geometry problems with complex numbers

Sample Problem:

(USA-TST-2006-6) Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

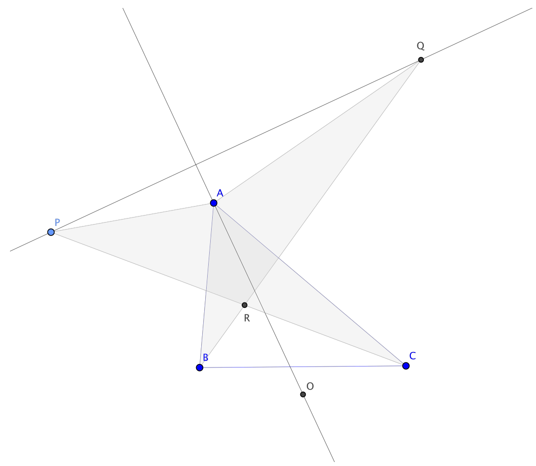


Chapter 12: Trigonometry

- Using trigonometry to solve olympiad geometry problems
- Trig Ceva, Law of Sines/Cosines, Ptolemy

Sample Problem:

(USA-TST-2006-6) Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.



MC50F Number Theory

Chapter 1: Divisibility & Euclidean Algorithm

- Using divisibility and Euclidean Algorithm to solve number theory problems
- Bezout, GCD/LCM, Fundamental Theorem of Arithmetic

Sample Problem:

(Classic) Suppose that an infinite sequence of positive integers $a_1, a_2, \ldots$ satisfies the property $\gcd(a_m, a_n) = \gcd(m, n)$ for all $m \neq n \geq 1$. Prove that $a_n = n$ for all $n \geq 1$.

Chapter 2: Modular Arithmetic

- Developing modular arithmetic techniques
- Fermat's Little Theorem, Euler's Totient Function, Wilson's Theorem

Sample Problem:

(Russia-2007) Let $p \geq 5$ be a prime. Show that the numbers $1!, 2!, \ldots, (p-1)!$ leave at least $\sqrt{p}$ different residues modulo p.

Chapter 3: Diophantine Equations

- Solving Diophantine Equations using divisibility/modular arithmetic

Sample Problem:

(USAJMO-2011-1) Find, with proof, all positive integers n for which $2^n + 12^n + 2011^n$ is a perfect square.

Chapter 4: Chinese Remainder Theorem

- Using the Chinese Remainder Theorem constructively to solve number theory problems

Sample Problem:

(USAMO-1991) Show that, for any fixed integer $n \geq 1$, the sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots \pmod{n}$$

is eventually constant. [The tower of exponents is defined by $a_1 = 2, a_{i+1} = 2^{a_i}$. Also $a_i \pmod{n}$ means the remainder which results from dividing a_i by n.]

Chapter 5: p-Adic Valuation and LTE

- Defining and proving basic properties about p-adic valuation
- Legendre's Formula and Lifting the Exponent

Sample Problem:

(BAMO-2018-4) Suppose that a, b, c are integers with the property that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

is an integer. Show that abc is a perfect cube.

Chapter 6: Order

- Defining and proving basic properties about order
- Thue's Lemma and Fermat's Sum of Two Squares Theorem

Sample Problem:

(ISL-2006-N5) Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

Chapter 7: Primitive Roots

- Defining and developing properties about primitive roots
- Number of primitive roots modulo n

Sample Problem:

(Romania-TST-2008) Find the greatest common divisor of the numbers

$$2^{561} - 2, 3^{561} - 3, \ldots, 561^{561} - 561.$$

Chapter 8: Quadratic Residues

- Defining quadratic residues/Legendre symbols
- Euler's Criterion
- Proving the Quadratic Reciprocity Law

Sample Problem:

(Taiwan-2000) Suppose that positive integers m and n satisfy $\varphi(5^m - 1) = 5^n - 1$. Show that $\gcd(m, n) > 1$.

Chapter 9: Integer Polynomials-I

- Using Schur's Theorem and Hensel's Lemma to solve number theory problems

Sample Problem:

(IMO-2006-5) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Chapter 10: Integer Polynomials-II

- Defining divisibility, GCD, and Bezout for polynomials
- Newton's Theorem, Lagrange Interpolation

Sample Problem:

(USA-TST-2010) Let P be a polynomial with integer coefficients such that $P(0) = 0$ and

$$\gcd(P(0), P(1), P(2), \ldots) = 1.$$

Show there are infinitely many n such that

$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$

Chapter 11: Arithmetic Functions

- Defining arithmetic functions and listing basic arithmetic functions
- Dirichlet Convolution and Möbius Inversion

Sample Problem:

(ISL-1989) Suppose the sequence $a_1, a_2, \dots$ satisfies

$$\sum_{d|n} a_d = 2^n.$$

Show that $n \mid a_n$ for all n.

Chapter 12: Advanced Diophantine Equations

- Using algebraic and number theoretic methods to solve general Diophantine equations

Sample Problem:

(Kolmogorov-Cup) Solve in integers a, b, c, n the equation $a^2 + b^2 + c^2 = 3n(ab + bc + ca)$.