Topics & Sample Problems

MC50F (USA(J)MO Fundamentals)

Contents

Part-I

MC50F Algebra

Chapter 1: Induction

- Using regular and strong induction to solve problems
- Applying different forms of induction (such as in multiple dimensions)

Sample Problem:

(Folklore) If $F_0 = 0, F_1 = 1$ and for all positive integers n, $F_{n+1} = F_n + F_{n-1}$, then prove

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}$$

for all nonnegative integers m, n.

Chapter 2: Sequences & Series

- Solving linear recurrences using generating functions
- Solving general problems involving sequences

Sample Problem:

(APMO-2015-3) A sequence of real numbers $a_0, a_1, ...$ is said to be good if the following three conditions hold. (i) The value of a_0 is a positive integer. (ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \frac{a_i}{a_i+2}$ (iii) There exists a positive integer k such that $a_k = 2014$.
Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, ...$ of real numbers with the property that $a_n = 2014$.

Chapter 3: Single variable polynomials

- Using Factor Theorem/Fundamental Theorem of Algebra to solve polynomial problems
- Lagrange Interpolation
- Derivative and Discriminant

Sample Problem:

(USA TSTST-2014-4) Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ such that: (i) both A and B have degree at most $d/2$ (ii) at most one of A and B is the zero polynomial. (iii) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x) + Q(x)B(x) = P(x)C(x)$.

Chapter 4: Symmetric polynomials

- Using symmetric polynomials and the Fundamental Theorem of Symmetric Polynomials to solve problems
- Vieta's Formulas and its applications

Sample Problem:

(USAMO-2014-1) Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Chapter 5: Factorization of Polynomials

- Proving polynomial irreducibility
- p-adic numbers, Newton polygons, and their applications toward polynomial irreducibility

Sample Problem:

(Romania TST-1998) Let n be a positive integer. Prove that $(x^2 + x)^{2^n} + 1$ is irreducible.

Chapter 6: The simplest inequalities

- Using the Trivial Inequality to solve inequality problems
- Sturm sequences and their applications to inequalities

Sample Problem:

(Centroamerican Olympiad-2014-3) Let a, b, c and d be real numbers such that no two of them are equal and

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4$$

and $ac = bd$. Find the maximum possible value of

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b}.$$

Chapter 7: Basic inequalities

- Using AM-GM and Cauchy-Schwarz to solve olympiad inequalities

Sample Problem:

(Turkey JBMO TST-2016-6) Prove that

$$(x^4 + y)(y^4 + z)(z^4 + x) \geq (x + y^2)(y + z^2)(z + x^2)$$

for all positive real numbers x, y, z satisfying $xyz \geq 1$.

Chapter 8: Famous Inequalities

- Using Rearrangement, Power Mean, Holder's Inequalities to solve olympiad inequalities

Sample Problem:

(ELMO-2013-2) Let a, b, c be positive reals satisfying $a + b + c = \sqrt[7]{a} + \sqrt[7]{b} + \sqrt[7]{c}$. Prove that $a^a b^b c^c \geq 1$.

Chapter 9: Convexity

- Convexity and its applications with Jensen's Inequality
- Using Jensen's Inequality and Karamata to solve olympiad inequalities

Sample Problem:

(ISL-2009-A2) Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$. Prove that:

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

Chapter 10: Symmetric, Homogeneous, Polynomial Inequalities

- Using Schur and Muirhead inequalities to solve olympiad inequalities
- Using the SOS and Equally Moving Variables technique to solve olympiad inequalities

Sample Problem:

(ISL-2008-A2) (i) If x, y and z are three real numbers, all different from 1, such that $xyz = 1$, then prove that $\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$. (With the $\sum$ sign for cyclic summation, this inequality could be rewritten as $\sum \frac{x^2}{(x-1)^2} \geq 1$.)
(ii) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

Chapter 11: Functional Equations-I

- Avoiding common mistakes while solving functional equations
- Using substitution to solve functional equations
- Solving functional equations with no nesting, such as Cauchy's functional equation

Sample Problem:

(USAMO-2014-2) Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

Chapter 12: Functional Equations-II

- Proving injectivity, surjectivity, and monotonicity and using them to solve functional equations

Sample Problem:

(ISL-2015-A2) Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

MC50F Counting

Chapter 1: Bijections

- Finding bijections and using to solve counting problems

Sample Problem:

(USAJMO-2013) Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be adjacent if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent). The filling is called a garden if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.

(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

Chapter 2: Invariants

- Finding invariants and monovariants and using them to prove counting problems

Sample Problem:

(USAMO-2015) Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k),(i,l),(j,k),(j,l)$ for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.
Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.
How many different non-equivalent ways can Steve pile the stones on the grid?

Chapter 3: Pigeonhole

- Using Pigeonhole and Infinite Pigeonhole to solve counting problems

Sample Problem:

(USAMO-2012) A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

Chapter 4: Induction

- Using induction and strong induction to solve problems, possibly in conjunction with recursion and infinite descent

Sample Problem:

(ISL-2006) We have $n \geq 2$ lamps $L_1, \ldots, L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows:

- if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off;
- otherwise, L_i is switched on.

Initially all the lamps are off except the leftmost one which is on.

(a) Prove that there are infinitely many integers n for which all the lamps will eventually be off.

(b) Prove that there are infinitely many integers n for which the lamps will never be all off.

Chapter 5: Recurrences

- Creating recurrences to solve counting problems
- Solving linear recurrences using characteristic polynomials

Sample Problem:

(USAMO-2013) For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

Chapter 6: Counting in Two Ways

- Counting a quantity in multiple ways to prove combinatorial identities and inequalities

Sample Problem:

(IMO-1998) In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

Chapter 7: Extrema

- Using the extremal principle to solve combinatorial geometry problems
- Using the extremal principle with respect to monovariants

Sample Problem:

(USA TST-2016-1) In a sports league, each team uses a set of at most t signature colors. A set S of teams is color-identifiable if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.
For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

Chapter 8: Graphs

- Defining basic graph terminology
- Developing and proving basic graph theorems

Sample Problem:

(Asian-Pacific Math Olympiad-2016) The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Chapter 9: Combinatorial Geometry

- Developing techniques to solve combinatorial geometry problems

Sample Problem:

(IMO-2013) A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement lines is good for a Colombian configuration if no line passes through any point of the configuration and no region contains points of both colors. Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Chapter 10: Games & Algorithms

- Discovering winning strategies to combinatorial games
- Sprague-Grundy number

Sample Problem:

(USAMO-1999) The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Chapter 11: Probabilistic Method & Expected Value

- Linearity of Expectation
- Using the Probabilistic Method to solve problems with nonconstructive solutions

Sample Problem:

(MOP-2007) An n by n square grid is filled with the numbers 1 through n, n times each. Prove that there exists a row or column with at least $\sqrt{n}$ different numbers.

Chapter 12: Generating Functions

- Using formal power series to solve combinatorial problems
- Generalized Binomial Theorem

Sample Problem:

(Erdos and Selfridge) Prove that the following statement holds iff n is a power of 2: there exist sequences $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ that the multisets $\{a_i + a_j | 1 \leq i < j \leq n\}$ and $\{b_i + b_j | 1 \leq i < j \leq n\}$ are the same.