Topics & Sample Problems

MC45F (AIME Advanced Fundamentals)

alphastar academy

Contents

Part-II

MC45F Geometry

Chapter 1: Angles

- Angles in circles and polygons; cyclic quadrilaterals
- Using angle chasing to solve problems

Sample Problem:

(CHMMC-2016-Individual-9) In quadrilateral $ABCD$, $AB = DB$ and $AD = BC$. If $m\angle ABD = 36^\circ$ and $m\angle BCD = 54^\circ$, find $m\angle ADC$ in degrees.

Chapter 2: Special Triangles

- equilateral, 30-60-90, 45-45-90, 15-75-90, 45-60-75, 36-72-72, and 18-72-90 triangles
- Pythagorean triples and Heronian scalenes

Sample Problem:

(Prasolov1 p101 q5.24) Points D and E divide sides AC and AB of an equilateral triangle ABC in the ratio of $AD : DC = BE : EA = 1 : 2$. Lines BD and CE meet at point O. Prove that $\angle AOC = 90^\circ$.

Chapter 3: Similarity

- Similarity and congruence conditions (SSS, ASA, SAA, AA and SAS similarity, not SSA)
- Angle bisector theorem

Sample Problem:

(PUMaC-2016-Geometry-7) Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10$, $AD = 24$, $XA = 17$, and $XB = 21$, then the area of ΔDEY can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

Chapter 4: Special Points

- Properties of the four triangle centers (centroid, orthocenter, incenter, circumcenter)
- The Euler line

Sample Problem:

(AIME-2010-I-15) In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

Chapter 5: Length-1

- Triangle inequality and Ravi substitution
- Pythagorean theorem and distance formula
- Mass points
- Ceva's Theorem, Menelaus' Theorem and Stewart's Theorem

Sample Problem:

Chapter 6: Length-2

- Length problems involving circles
- Power of a point
- Radical axis and radical center
- Ptolemy's theorem

Sample Problem:

(PUMaC-2014-Geometry-4) Consider the cyclic quadrilateral with sides 1, 4, 8, 7 in that order. What is its circumdiameter? Let the answer be of the form $a\sqrt{b} + c$, for b square free. Find $a + b + c$.

Chapter 7: Area-1

- Triangle area formulas
- Special quadrilateral area formulas such as Brahmagupta's formula

Sample Problem:

Chapter 8: Area-2

- Area problems involving length ratios

Sample Problem:

(HMMT Feb-2004-Geometry-10) Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, Z lie on XZ in that order. Find the value of $(PY + YZ)(QY + XY)$.

Chapter 9: Trigonometry

- Definitions of trigonometric functions, basic trig identities, sum and difference formulas
- Law of sines, law of cosines, ratio lemma
- Trigonometric Ceva

Sample Problem:

(AIME-2018-I-13) Let ΔABC have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of ΔABX and ΔACX, respectively. Find the minimum possible area of ΔAI_1I_2 as X varies along BC.

Chapter 10: Analytic Geometry

- Distance formulas (between two points, point & line)
- The slope and the equation of a line (slope-intercept and point slope)
- Reflections over lines
- Equation of circles
- Shoelace formula and Pick's theorem

Sample Problem:

(HMMT Feb-2010-Guts-13) A triangle in the xy-plane is such that when projected onto the x-axis, y-axis, and the line $y = x$, the results are line segments whose endpoints are $(1, 0)$ and $(5, 0)$, $(0, 8)$ and $(0, 13)$, and $(5, 5)$ and $(7.5, 7.5)$, respectively. What is the triangle's area?

Chapter 11: Complex Numbers

- Introduction to radians, Euler's formula
- Various representations of complex numbers
- Magnitude, argument and distance in complex plane
- Rotations, colinearity, perpendicularity

Sample Problem:

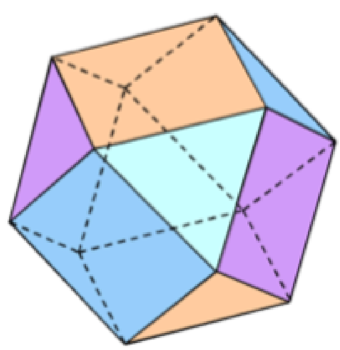
(AIME-2012-I-14) Complex numbers a, b, and c are zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

Chapter 12: 3D

- Platonic solids, spheres, cylinders, cones
- Distance formula, point-to-plane formula, Euler characteristic
- Using cross-sections and 2D properties to solve 3D problems

Sample Problem:

(PUMaC-2010-Geometry-5) A cuboctahedron is a solid with 6 square faces and 8 equilateral triangle faces, with each edge adjacent to both a square and a triangle (see picture). Suppose the ratio of the volume of an octahedron to a cuboctahedron with the same side length is r. Find $100r^2$.



MC45F Number Theory

Chapter 1: Number Bases

- Non-decimal bases
- Legendre's formula

Sample Problem:

(AIME-2010-I-10) Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \leq a_i \leq 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

Chapter 2: Primes & Prime Factorization

- Definitions of primes and Euclid's Lemma
- Fundamental Theorem of Arithmetic

Sample Problem:

Chapter 3: Divisibility

- Divisibility rules
- p-adic valuation
- Lifting the exponent

Sample Problem:

(AIME-2006-II-14) Let S_n be the sum of the reciprocals of the nonzero digits of the integers from 1 to 10^n, inclusive. Find the smallest positive integer n for which S_n is an integer.

Chapter 4: Multiplicative Functions

- Problems involving multiplicative functions, such as Divisor function, Sigma function, Totient function
- Properties of φ function

Sample Problem:

(AIME-2016-II-11) For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Chapter 5: Factoring Techniques

- Difference of squares and arbitrary powers, sum of cubes and odd powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity

Sample Problem:

(CHMMC-2012 Fall-Individual-8) Find two pairs of positive integers (a, b) with $a > b$ such that

$$a^2 + b^2 = 40501.$$

Chapter 6: GCD & LCM

- Greatest common divisor, least common multiple
- Euclidean algorithm and its applications
- Bezout's identity

Sample Problem:

(CHMMC-2010 Fall-Team-5) The three positive integers a, b, c satisfy the equalities $\gcd(ab, c^2) = 20$, $\gcd(ac, b^2) = 18$, and $\gcd(bc, a^2) = 75$. Compute the minimum possible value of $a + b + c$.

Chapter 7: Modular Arithmetic

- Properties of modulo
- Modular inverses
- Using binomial theorem to find remainders

- Solving AIME level problems using modular arithmetic

Sample Problem:

(AIME-2012-I-15) There are n mathematicians seated around a circular table with n seats numbered 1, 2, 3, ..., n in clockwise order. After a break they again sit around the table. The mathematicians note that there is a positive integer a such that

1. for each k, the mathematician who was seated in seat k before the break is seated in seat ka after the break (where seat $i + n$ is seat i);
2. for every pair of mathematicians, the number of mathematicians sitting between them after the break, counting in both the clockwise and the counterclockwise directions, is different from either of the number of mathematicians sitting between them before the break.

Find the number of possible values of n with $1 < n < 1000$.

Chapter 8: Fermat's Little Theorem & Euler Theorem

- Fermat's little theorem
- Euler's totient theorem
- Wilson's theorem

Sample Problem:

(HMMT Feb-2010-Guts-29) Compute the remainder when

$$\sum_{k=1}^{30303} k^k$$

is divided by 101.

Chapter 9: Chinese Remainder Theorem

- Chinese remainder theorem (CRT)
- Computing solutions to CRT, using CRT backwards

Sample Problem:

(AIME-2012-I-10) Let $\mathcal{S}$ be the set of all perfect squares whose rightmost three digits in base 10 are 256. Let $\mathcal{T}$ be the set of all numbers of the form $\frac{x-256}{1000}$, where x is in $\mathcal{S}$. In other words, $\mathcal{T}$ is the set of numbers that result when the last three digits of each number in $\mathcal{S}$ are truncated. Find the remainder when the tenth smallest element of $\mathcal{T}$ is divided by 1000.

Chapter 10: Degree

- Definition and properties of order modulo m

Sample Problem:

Chapter 11: Primitive Roots

- Definition of primitive roots
- Primitive root theorem
- Finding number of equations of modular equations using primitive roots

Sample Problem:

(CHMMC-2010 Winter-Individual-15) Compute the number of primes p less than 100 such that p divides $n^2 + n + 1$ for some integer n.

Chapter 12: Diophantine Equations

- General strategies for solving AIME level diophantine equations
- Chicken McNugget Theorem
- Computing Pythagorean triples
- Fermat's Last Theorem (optional)

Sample Problem:

(AIME-2008-II-15) Find the largest integer n satisfying the following conditions:
(i) n^2 can be expressed as the difference of two consecutive cubes;
(ii) $2n + 79$ is a perfect square.