Topics & Sample Problems

MC45F (AIME Advanced Fundamentals)

Contents

Part-I

MC45F Algebra

Chapter 1: Word Problems

- Developing logical analysis and boost creative thinking by solving word problems.
- Converting word problems into mathematical equations and solving AIME level system of equations.

Sample Problem:

(HMMT Feb-2012-Guts-17) Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William?s reverses direction at the same speed). Initially, Mark's orb is moving at 1/1000 meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

Chapter 2: Sequences & Series

- Finding patterns in sequences by looking at small cases.
- Using trig substitution and invariance in sequence problems.
- Understanding recurrence relations and solving linear recurrences.
- Finding closed-form formulas for sequences.

Sample Problem:

Chapter 3: Functions-1

- Solving equations that involve special functions such as floor, ceiling and absolute value

- Counting functions using information about its domain and range

Sample Problem:

Chapter 4: Functions-2

- Solving functional equations using substitution, injectivity, and surjectivity, symmetry

Sample Problem:

(HMMT Nov-2015-Guts-26) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y. If $f(2) = 0$, compute $f(2015)$.

Chapter 5: Polynomials-1

- Finding roots of some cubic, quartic, and higher degree polynomials using substitution, binomial theorem
- Vieta's theorem and its applications
- Using techniques such as long division, factor theorem and rational root theorem when finding roots of higher degree polynomials

Sample Problem:

(PUMaC-2010-Algebra-5) Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots of f are given by α, β, and γ, find

$$\left(\frac{1}{\alpha - 2}\right)^2 + \left(\frac{1}{\beta - 2}\right)^2 + \left(\frac{1}{\gamma - 2}\right)^2.$$

Chapter 6: Polynomials-2

- Solving polynomial equations using Lagrange interpolation and Finite differences

Sample Problem:

Chapter 7: Logarithm

- Solving AIME level problems involving logarithms and natural logarithm

Sample Problem:

(AIME-2006-I-9) The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

Chapter 8: Trigonometry

- Solving algebra problems using trig substitution, trig identities and formulas

Sample Problem:

(SMT-2014-Algebra Tiebreaker-3) Compute $\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}}$.

Chapter 9: Complex Numbers-1

- Having a deep knowledge of complex numbers, finding roots of polynomials with complex roots
- Algebraic operations involving complex numbers and complex plane
- Problem solving techniques using Euler's formula and de Moivre's formula

Sample Problem:

(AIME-2013-I-14) For $\pi \leq \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \ldots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \ldots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Chapter 10: Complex Numbers-2

- Finding roots of unity and using algebraic operations on roots of unity to solve problems

Sample Problem:

(HMMT Feb-2006-Guts-21) Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

Chapter 11: System of Equations

- Solving system of equations using polynomials, substitutions and symmetry

Sample Problem:

Chapter 12: Inequalities

- Finding minimum/maximum of algebraic expressions using elementary properties of inequalities, such as transitivity and algebraic operations on inequalities
- Arithmetic Mean - Geometric Mean (AM-GM) Inequality
- Cauchy-Schwarz Inequality
- Some advanced inequalities such as Rearrangement Inequality, Jensen's Inequality and weighted AM-GM Inequality

Sample Problem:

MC45F Counting

Chapter 1: Basic Counting Techniques

- Solving counting problems using techniques such as casework and complementary counting

Sample Problem:

Chapter 2: Counting Sets & PIE

- Solving counting problems using the Principle of Inclusion and Exclusion (PIE)

Sample Problem:

Chapter 3: Path Counting & Bijections

- Solving counting problems using bijections
- Solving path-counting problems

Sample Problem:

Chapter 4: Stars and Bars

- Solving counting problems using the Stars and Bars method

Sample Problem:

Chapter 5: Binomial

- Solving counting problems involving binomials and multinomials
- Binomial identities such as Hockey-Stick Identity and Vandermonde's Identity

Sample Problem:

Chapter 6: Counting with Recursion

- Identifying which counting problems can be solved using recursions
- Finding and solving recursions

Sample Problem:

Chapter 7: Probability

- Solving difficult probability problems
- Conditional probability and Bayes' Theorem
- Geometric probability

Sample Problem:

Chapter 8: Expected Value

- Random variables, expected value and variance
- Solving geometry problems involving expected values
- Properties of expectation, such as linearity of expectation

Sample Problem:

Chapter 9: Markov Chains

- Solving problems using Markov chains and state diagrams

Sample Problem:

Chapter 10: Geometric Counting

- Solving counting problems related to geometric objects
- Euler's Formula

Sample Problem:

Chapter 11: Generating Functions

- Using generating functions to turn counting problems into algebra
- Counting number of partitions

Sample Problem:

Chapter 12: Catalan Numbers

- Using Catalan numbers to solve counting problems

Sample Problem:

(HMMT Feb-2007-Combinatorics-10) A subset S of the nonnegative integers is called *supported* if it contains 0, and $k+8, k+9 \in S$ for all $k \in S$. How many supported sets are there?