# Topics & Sample Problems

# MC40F (AIME Basic Fundamentals)

alphastar academy

# Contents

# Part-II

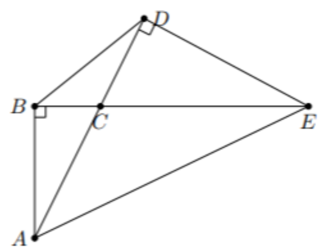
# MC40F Geometry

## Chapter 1: Angles

- Angles in circles and polygons; cyclic quadrilaterals
- Using angle chasing to solve problems

**Sample Problem:**

(CHMMC-2012 Fall-Team-4) Consider the figure below, not drawn to scale.



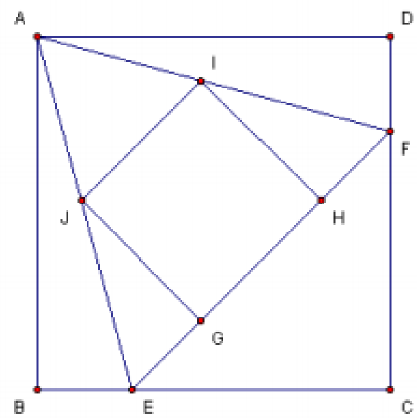

In this figure, assume that $AB \perp BE$ and $AD \perp DE$. Also, let $AB = \sqrt{6}$ and $\angle BED = \frac{\pi}{6}$. Find $AC$.

## Chapter 2: Special Triangles

- equilateral, 30-60-90, 45-45-90, 15-75-90, 45-60-75, 36-72-72, and 18-72-90 triangles
- Pythagorean triples and Heronian scalenes

**Sample Problem:**

(SMT-2011-Team-1) Let $ABCD$ be a unit square. The point $E$ lies on $BC$ and $F$ lies on $CD$. $\triangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\triangle AEF$ so that $GH$ is on $EF$. Compute the area of $GHIJ$.




## Chapter 3: Similarity

- Similarity and congruence conditions (SSS, ASA, SAA, AA and SAS similarity, not SSA)
- Angle bisector theorem

**Sample Problem:**

(AMC12-2002-A23) In triangle $ABC$, side $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ meet in point $D$, and $\overline{BD}$ bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle $ABD$?

**(A)** 14 **(B)** 21 **(C)** 28 **(D)** $14\sqrt{5}$ **(E)** $28\sqrt{5}$

## Chapter 4: Special Points

- Properties of the four triangle centers (centroid, orthocenter, incenter, circumcenter)
- The Euler line

**Sample Problem:**

(AIME-2016-I-6) In $\triangle ABC$ let $I$ be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect $AB$ at $L$. The line through $C$ and $L$ intersects the circumscribed circle of $\triangle ABC$ at the two points $C$ and $D$. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

## Chapter 5: Length-1

- Triangle inequality and Ravi substitution
- Pythagorean theorem and distance formula
- Mass points
- Ceva's Theorem, Menelaus' Theorem and Stewart's Theorem

**Sample Problem:**

## Chapter 6: Length-2

- Length problems involving circles
- Power of a point
- Radical axis and radical center
- Ptolemy's theorem

**Sample Problem:**

(PUMaC-2010-Geometry-6) A semicircle is folded along a chord $AN$ and intersects its diameter $MN$ at $B$. Given that $MB : BN = 2 : 3$ and $MN = 10$, if $AN = x$, find $x^2$.

## Chapter 7: Area-1

- Triangle area formulas
- Special quadrilateral area formulas such as Brahmagupta's formula

**Sample Problem:**

(AIME-2016-II-7) Squares $ABCD$ and $EFGH$ have a common center and $\overline{AB}||\overline{EF}$. The area of $ABCD$ is 2016, and the area of $EFGH$ is a smaller positive integer. Square $IJKL$ is constructed so that each of its vertices lies on a side of $ABCD$ and each vertex of $EFGH$ lies on a side of $IJKL$. Find the difference between the largest and smallest positive integer values for the area of $IJKL$.

## Chapter 8: Area-2

- Area problems involving length ratios

**Sample Problem:**

## Chapter 9: Trigonometry

- Definitions of trigonometric functions, basic trig identities, sum and difference formulas
- Law of sines, law of cosines, ratio lemma
- Trigonometric Ceva

**Sample Problem:**

(AMC12-2018-A23) In $\triangle PAT, \angle P = 36^\circ, \angle A = 56^\circ$, and $PA = 10$. Points $U$ and $G$ lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let $M$ and $N$ be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines $MN$ and $PA$?

**(A)** 76 **(B)** 77 **(C)** 78 **(D)** 79 **(E)** 80

## Chapter 10: Analytic Geometry

- Distance formulas (between two points, point & line)
- The slope and the equation of a line (slope-intercept and point slope)
- Reflections over lines
- Equation of circles
- Shoelace formula and Pick's theorem

**Sample Problem:**

## Chapter 11: Complex Numbers

- Introduction to radians, Euler's formula
- Various representations of complex numbers
- Magnitude, argument and distance in complex plane
- Rotations, colinearity, perpendicularity

**Sample Problem:**

(Math Day at the Beach-2010-Team-6) Let $z_1, z_2, \ldots, z_{10}$ be complex numbers that form a regular decagon (10-sided polygon) in the complex plane, with that decagon inscribed in a circle of radius $\sqrt[5]{7}$ centered at 2. At least one of the $z_k$ is real. Compute the product $z_1 z_2 \cdots z_{10}$.

## Chapter 12: 3D

- Platonic solids, spheres, cylinders, cones
- Distance formula, point-to-plane formula, Euler characteristic
- Using cross-sections and 2D properties to solve 3D problems

**Sample Problem:**

# MC40F Number Theory

## Chapter 1: Number Bases

- Non-decimal bases
- Legendre's formula

**Sample Problem:**

(Folklore) What is the $200^{\text{th}}$ smallest positive integer that can be written as the sum of distinct powers of 3?

## Chapter 2: Primes & Prime Factorization

- Definitions of primes and Euclid's Lemma
- Fundamental Theorem of Arithmetic

**Sample Problem:**

(AIME-2006-I-4) Let $N$ be the number of consecutive 0's at the right end of the decimal representation of the product $1!2!3!4!\ldots99!100!$. Find the remainder when $N$ is divided by 1000.

## Chapter 3: Divisibility

- Divisibility rules
- p-adic valuation
- Lifting the exponent

**Sample Problem:**

## Chapter 4: Multiplicative Functions

- Problems involving multiplicative functions, such as Divisor function, Sigma function, Totient function
- Properties of $\varphi$ function

**Sample Problem:**

(BMT-2012-Tournament-Round7-P3) Let $\varphi$ be the Euler totient function, and let $S = \{x \mid \frac{x}{\varphi(x)} = 3\}$. What is $\sum_{x \in S} \frac{1}{x}$? Express your answer as a common fraction in reduced form.

## Chapter 5: Factoring Techniques

- Difference of squares and arbitrary powers, sum of cubes and odd powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity

**Sample Problem:**

(HMMT Nov-2008-Guts-31) Find the sum of all primes $p$ for which there exists a prime $q$ such that $p^2 + pq + q^2$ is a square.

## Chapter 6: GCD & LCM

- Greatest common divisor, least common multiple
- Euclidean algorithm and its applications
- Bezout's identity

**Sample Problem:**

## Chapter 7: Modular Arithmetic

- Properties of modulo
- Modular inverses
- Using binomial theorem to find remainders
- Solving AIME level problems using modular arithmetic

**Sample Problem:**

(AMC12-2014-B23) The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when $S$ is divided by 2017?

**(A)** 32 **(B)** 684 **(C)** 1024 **(D)** 1576 **(E)** 2016

## Chapter 8: Fermat's Little Theorem & Euler Theorem

- Fermat's little theorem
- Euler's totient theorem
- Wilson's theorem

**Sample Problem:**

(ARML-2002-Individual-6) Let $a$ be the integer such that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{22} + \frac{1}{23} = \frac{a}{23!}$. Compute the remainder when $a$ is divided by 13.

## Chapter 9: Chinese Remainder Theorem

- Chinese remainder theorem (CRT)
- Computing solutions to CRT, using CRT backwards

**Sample Problem:**

## Chapter 10: Degree

- Definition and properties of order modulo m

**Sample Problem:**

(CHMMC-2014-Individual-7) A robot is shuffling a 9 card deck. Being very well machined, it does every shuffle in exactly the same way: it splits the deck into two piles, one containing the 5 cards from the bottom of the deck and the other with the 4 cards from the top. It then interleaves the cards from the two piles, starting with a card from the bottom of the larger pile at the bottom of the new deck, and then alternating cards from the two piles while maintaining the relative order of each pile. The top card of the new deck will be the top card of the bottom pile.
The robot repeats this shuffling procedure a total of $n$ times, and notices that the cards are in the same order as they were when it started shuffling. What is the smallest possible value of $n$?

## Chapter 11: Primitive Roots

- Definition of primitive roots
- Primitive root theorem
- Finding number of equations of modular equations using primitive roots

**Sample Problem:**

(Kevin Li) How many primitive roots does 1458 have?

## Chapter 12: Diophantine Equations

- General strategies for solving AIME level diophantine equations
- Chicken McNugget Theorem
- Computing Pythagorean triples
- Fermat's Last Theorem (optional)

**Sample Problem:**

(PUMaC-2014-Number Theory-5) Find the number of pairs of integer solutions $(x, y)$ that satisfies the equation

$$(x-y+2)(x-y-2) = -(x-2)(y-2).$$