Topics & Sample Problems

MC40F (AIME Basic Fundamentals)

Contents

Part-I

MC40F Algebra

Chapter 1: Word Problems

- Developing logical analysis and boost creative thinking by solving word problems.
- Converting word problems into mathematical equations and solving AIME level system of equations.

Sample Problem:

(PUMaC-2012-Team-2.3.1) For some reason, people in math problems like to paint houses. Alice can paint a house in one hour. Bob can paint a house in six hours. If they work together, it takes them seven hours to paint a house. You might be thinking "What? That's not right!" but I did not make a mistake.
When Alice and Bob work together, they get distracted very easily and simultaneously send text messages to each other. When they are texting, they are not getting any work done. When they are not texting, they are painting at their normal speeds (as if they were working alone). Carl, the owner of the house decides to check up on their work. He randomly picks a time during the seven hours. The probability that they are texting during that time can be written as r/s, where r and s are integers and $\gcd(r,s) = 1$. What is $r + s$?

Chapter 2: Sequences & Series

- Finding patterns in sequences by looking at small cases.
- Using trig substitution and invariance in sequence problems.
- Understanding recurrence relations and solving linear recurrences.
- Finding closed-form formulas for sequences.

Sample Problem:

(AMC12-2016-B25) The sequence (a_n) is defined recursively by $a_0 = 1, a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Chapter 3: Functions-1

- Solving equations that involve special functions such as floor, ceiling and absolute value
- Counting functions using information about its domain and range

Sample Problem:

Chapter 4: Functions-2

- Solving functional equations using substitution, injectivity, and surjectivity, symmetry

Sample Problem:

Chapter 5: Polynomials-1

- Finding roots of some cubic, quartic, and higher degree polynomials using substitution, binomial theorem
- Vieta's theorem and its applications
- Using techniques such as long division, factor theorem and rational root theorem when finding roots of higher degree polynomials

Sample Problem:

(Iurie Boreico) A rectangular box has volume equal to 6, surface area equal to 30, and diagonal equal to $\sqrt{34}$. The largest dimension of the box is $a + \sqrt{b}$ where a, b are positive integers. Find $a + b$.

Chapter 6: Polynomials-2

- Solving polynomial equations using Lagrange interpolation and Finite differences

Sample Problem:

Chapter 7: Logarithm

- Solving AIME level problems involving logarithms and natural logarithm

Sample Problem:

Chapter 8: Trigonometry

- Solving algebra problems using trig substitution, trig identities and formulas

Sample Problem:

Chapter 9: Complex Numbers-1

- Having a deep knowledge of complex numbers, finding roots of polynomials with complex roots
- Algebraic operations involving complex numbers and complex plane
- Problem solving techniques using Euler's formula and de Moivre's formula

Sample Problem:

(AIME-2016-I-7) For integers a and b consider the complex number

$$\frac{\sqrt{ab+2016}}{ab+100}-\left(\frac{\sqrt{|a+b|}}{ab+100}\right)i$$

Find the number of ordered pairs of integers (a,b) such that this complex number is a real number.

Chapter 10: Complex Numbers-2

- Finding roots of unity and using algebraic operations on roots of unity to solve problems

Sample Problem:

(HMMT Feb-2008-Algebra-6) A root of unity is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2+az+b=0$ for some integers a and b.

Chapter 11: System of Equations

- Solving system of equations using polynomials, substitutions and symmetry

Sample Problem:

(AIME-2000-I-7) Suppose that x, y, and z are three positive numbers that satisfy the equations $xyz=1$, $x+\frac{1}{z}=5$, and $y+\frac{1}{x}=29$. Then $z+\frac{1}{y}=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Chapter 12: Inequalities

- Finding minimum/maximum of algebraic expressions using elementary properties of inequalities, such as transitivity and algebraic operations on inequalities
- Arithmetic Mean - Geometric Mean (AM-GM) Inequality
- Cauchy-Schwarz Inequality
- Some advanced inequalities such as Rearrangement Inequality, Jensen's Inequality and weighted AM-GM Inequality

Sample Problem:

(SMT-2018-Algebra Tiebreaker-1) If a, b, c are real numbers with $a - b = 4$, find the maximum value of $ac + bc - c^2 - ab$.

MC40F Counting

Chapter 1: Basic Counting Techniques

- Solving counting problems using techniques such as casework and complementary counting

Sample Problem:

(AIME-2008-I-7) Let S_i be the set of all integers n such that $100i \leq n < 100(i+1)$. For example, S_4 is the set $400, 401, 402, \ldots, 499$. How many of the sets $S_0, S_1, S_2, \ldots, S_{999}$ do not contain a perfect square?

Chapter 2: Counting Sets & PIE

- Solving counting problems using the Principle of Inclusion and Exclusion (PIE)

Sample Problem:

(Kevin Liu) How many functions $f : \{1, 2, \ldots, 6\} \to \{1, 2, \ldots, 6\}$ are there such that $\{1, 2, 3\}$ is a subset of the range of f?

Chapter 3: Path Counting & Bijections

- Solving counting problems using bijections
- Solving path-counting problems

Sample Problem:

(Brice Huang) How many ways are there to write 10 as the sum of any number of positive integers if different orderings of the same sum are distinguishable?

Chapter 4: Stars and Bars

- Solving counting problems using the Stars and Bars method

Sample Problem:

(PUMaC-2014-Team-5) How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a + b + c = 103$?

Chapter 5: Binomial

- Solving counting problems involving binomials and multinomials
- Binomial identities such as Hockey-Stick Identity and Vandermonde's Identity

Sample Problem:

(AIME-2000-II-7) Given that

$$\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!}$$

find the greatest integer that is less than $\frac{N}{100}$.

Chapter 6: Counting with Recursion

- Identifying which counting problems can be solved using recursions
- Finding and solving recursions

Sample Problem:

Chapter 7: Probability

- Solving difficult probability problems
- Conditional probability and Bayes' Theorem
- Geometric probability

Sample Problem:

(AIME-2014-II-6) Charles has two six-sided dice. One of the die is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$ and each of the other five sides has probability $\frac{1}{15}$. Charles chooses one of the two dice at random and rolls it three times. Given that the first two rolls are both sixes, the probability that the third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Chapter 8: Expected Value

- Random variables, expected value and variance
- Solving geometry problems involving expected values
- Properties of expectation, such as linearity of expectation

Sample Problem:

(HMMT Feb-2010-Guts-9) Indecisive Andy starts out at the midpoint of the 1-unit-long segment $\overline{HT}$. He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H, and if the coin is tails, he moves halfway towards endpoint T. After his 2010 moves, what is the expected distance between Andy and the midpoint of $\overline{HT}$? Express your answer as a decimal to the nearest hundredth.

Chapter 9: Markov Chains

- Solving problems using Markov chains and state diagrams

Sample Problem:

(PUMaC-2010-Combinatorics-4) Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares and there are \$5 bills in the top right and bottom left squares, as shown below.

\$1				\$5
	E			
\$5				\$1

Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 10: Geometric Counting

- Solving counting problems related to geometric objects
- Euler's Formula

Sample Problem:

Chapter 11: Generating Functions

- Using generating functions to turn counting problems into algebra
- Counting number of partitions

Sample Problem:

(Christopher Shao) Find the number of solutions to $a + b + c = 4$ if $-3 \leq a \leq -1, 0 \leq b \leq 2, 3 \leq c \leq 5$ and a, b, and c are integers.

Chapter 12: Catalan Numbers

- Using Catalan numbers to solve counting problems

Sample Problem: