Topics & Sample Problems

MC35F (AMC 10/12 Advanced Fundamentals)

Contents

Part-II

MC35F Geometry

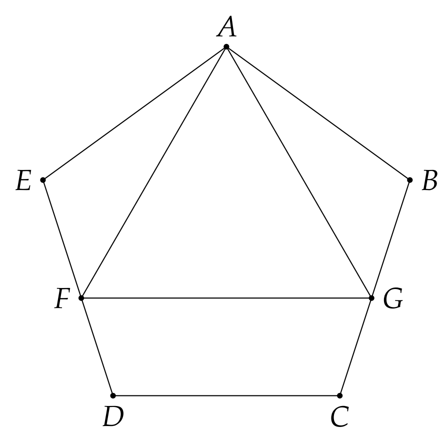
Chapter 1: Angles

- Angles (review)
- Inscribed angles in a circle, cyclic quadrilaterals

Sample Problem:

(Math Day at the Beach-2012-Individual-19) The figure below contains a regular pentagon and an equilateral triangle. Let $a < b < c < d < e$ be all the different measures of all of the angles in the picture. Compute

$$\frac{b}{a} + \frac{e}{d} + \frac{d}{b}.$$



Chapter 2: Special Triangles

- 30-60-90, 45-45-90, and 15-75-90 triangles
- Pythagorean triples

Sample Problem:

(BMT-2016-Geometry-4) ABC is an equilateral triangle, and $ADEF$ is a square. If D lies on side AB and E lies on side BC, what is the ratio of the area of the equilateral triangle to the area of the square?

Chapter 3: Similarity

- Similarity/congruence axioms (SSS, SAS, ASA, AA similarity)
- Power of a point, angle bisector theorem

Sample Problem:

(AMC10-2016-A19) In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$, where the greatest common factor of r, s, and t is 1. What is $r + s + t$?

(A) 7 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 20

Chapter 4: Special Points

- Special points of a triangle (centroid, incenter, circumcenter, orthocenter)

Sample Problem:

(AMC12-2012-A18) Triangle ABC has $AB = 27, AC = 26$, and $BC = 25$. Let I denote the intersection of the internal angle bisectors $\triangle ABC$. What is BI?

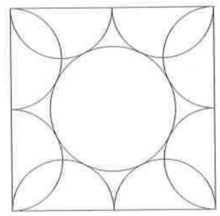
(A) 15 **(B)** $5 + \sqrt{26} + 3\sqrt{3}$ **(C)** $3\sqrt{26}$ **(D)** $\frac{2}{3}\sqrt{546}$ **(E)** $9\sqrt{3}$

Chapter 5: Length-1

- Triangle inequality, Ravi substitution
- Pythagorean theorem, distance formula
- Stewart's theorem

Sample Problem:

(AMC10-2014-B22) Eight semicircles line the inside of a square with side length 2 as shown. What is the radius of the circle tangent to all these semicircles?



(A) $\frac{1+\sqrt{2}}{4}$ **(B)** $\frac{\sqrt{5}-1}{2}$ **(C)** $\frac{\sqrt{3}+1}{4}$ **(D)** $\frac{2\sqrt{3}}{5}$ **(E)** $\frac{\sqrt{5}}{3}$

Chapter 6: Length-2

- Mass points considering levers/torque
- Ceva's theorem, Menelaus' theorem

Sample Problem:

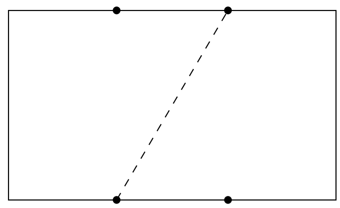
(Challenging Pr in Geo p37 q8) In right $\triangle ABC$, P and Q are on $\overline{BC}$ and $\overline{AC}$, respectively, such that $CP = CQ = 2$. Through the point of intersection, R, of $\overline{AP}$ and $\overline{BQ}$, a line is drawn also passing through C and meeting $\overline{AB}$ at S. $\overline{PQ}$ extended meets line AB at T. If the hypotenuse $AB = 10$ and $AC = 8$, find TS.

Chapter 7: Area-1

- Areas of simple shapes (triangle, certain quadrilaterals)
- Triangle area formulas (Heron's formula, A = rs, A = abc/4R, (ab sin C)/2

Sample Problem:

(AMC10-2014-A23) A rectangular piece of paper whose length is $\sqrt{3}$ times the width has area A. The paper is divided into three equal sections along opposite lengths, and then a dotted line is drawn from the first divider to the second divider on the opposite side as shown. The paper is then folded flat along this dotted line to create a new shape with area B. What is the ratio $B : A$?



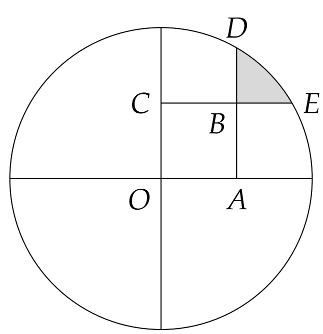
(A) 1:2 **(B)** 3:5 **(C)** 2:3 **(D)** 3:4 **(E)** 4:5

Chapter 8: Area-2

- Area formula for a circle, sector
- Brahmagupta's formula
- Area of more complicated shapes involving circles and/or other polygons

Sample Problem:

(AMC10-2006-B19) A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E?



(A) $\frac{\pi}{3} + 1 - \sqrt{3}$ **(B)** $\frac{\pi}{2}(2 - \sqrt{3})$ **(C)** $\pi(2 - \sqrt{3})$ **(D)** $\frac{\pi}{6} + \frac{\sqrt{3}+1}{2}$ **(E)** $\frac{\pi}{3} - 1 + \sqrt{3}$

Chapter 9: Trigonometry-1

- Definitions of sin, cos, tan, as well as csc, sec, cot
- The unit circle
- Basic trig identities, Sum and difference formulas for sin, cos, tan (e.g. sin(a+b))

Sample Problem:

(PPP Vol3 p21 q17) Suppose $\sqrt{6}\sin x + \sqrt{2}\cos x = 2$, where $0 \le x \le 2\pi$. Find all possible values of $\cos 2x$.

Chapter 10: Trigonometry-2

- Law of sines
- Law of cosines
- Ratio lemma, trig Ceva's theorem
- Solving algebra problems by trig substitution

Sample Problem:

(Hong Kong MC-2002-14) In $\triangle ABC$, $\angle ACB = 3\angle BAC$, $BC = 5$, $AB = 11$. Find AC.

Chapter 11: Analytic Geometry

- Slope, equation of a line using slope-intercept or point-slope form, distance and midpoint formulas
- Reflections over lines in the coordinate plan
- Equation of a circle
- Shoelace formula

Sample Problem:

(AMC10-2014-A18) A square in the coordinate plane has vertices whose y-coordinates are $0, 1, 4$, and 5. What is the area of the square?

(A) 16 **(B)** 17 **(C)** 25 **(D)** 26 **(E)** 27

Chapter 12: 3D

- Distance formula in 3D
- Area/volume of various 3D shapes (cube, prisms, cylinders, cones, spheres)
- Common 3D solids
- Euler's polyhedral formula

Sample Problem:

(AMC10-2012-B23) A solid tetrahedron is sliced off a solid wooden unit cube by a plane passing through two nonadjacent vertices on one face and one vertex on the opposite face not adjacent to either of the first two vertices. The tetrahedron is discarded and the remaining portion of the cube is placed on a table with the cut surface face down. What is the height of this object?

(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{2\sqrt{2}}{3}$ **(C)** 1 **(D)** $\frac{2\sqrt{3}}{3}$ **(E)** $\sqrt{2}$

MC35F Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences
- Sum of squares and sum of cubes formula
- Sigma notation

Sample Problem:

(AMC10-2016-B18) In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 7

Chapter 2: Primes & Prime Factorization

- Definition of divisibility ($a \mid b$)
- Prime and composite numbers, Euclid's proof of the infinitude of primes
- Fundamental theorem of arithmetic
- Legendre's formula
- Prime number theorem, prime checking algorithms (optional)

Sample Problem:

(AIME-2006-II-3) Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

Chapter 3: Divisibility

- Divisibility by numbers 2-11 inclusive
- Various problems involving divisibility, prime factorization, etc.

Sample Problem:

(ARML-2006-Individual-4) Compute the four digit positive integer N whose square root is three times the sum of the digits of N.

Chapter 4: Number & Sum of Divisors

- General formula for the number and sum of divisors of a positive integer n, given its prime factorization
- Perfect, abundant, and deficient numbers (optional)

Sample Problem:

(ARML-2014-Individual-6) Compute the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.

Chapter 5: Factoring Techniques

- Difference of squares
- Sum and difference of cubes; sum and difference of n-th powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity
- Informal definition of an irreducible polynomial over the integers (e.g. $x^2 + y^2$)

Sample Problem:

(SMT-2018-General-18) How many integer pairs (a, b) satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{2018}$?

Chapter 6: Number Bases

- Conversion between different number bases (emphasis on base 2, 8, 10, and 16)
- Arithmetic in different bases
- Fast base conversion (e.g. binary to hexadecimal)

Sample Problem:

(SMT-2012-Advanced Topics-2) Find the sum of all integers x, $x \geq 3$, such that

$$201020112012_x$$

(that is, 201020112012 interpreted as a base x number) is divisible by $x - 1$.

Chapter 7: GCD & LCM

- Definition of relatively prime
- Computing the GCD and LCM using the prime factorization
- Computing the GCD of two numbers using the Euclidean algorithm

Sample Problem:

(HMMT Nov-2015-Guts-15) Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution."

Chapter 8: Modular Arithmetic

- Basic properties of the modulo (reflexive, symmetric, transitive, etc.)
- Proof of divisibility rules using modular arithmetic
- Modular inverses
- More advanced modulo calculations involving basic operations

Sample Problem:

(AIME-2010-I-2) Find the remainder when $9 \cdot 99 \cdot 999 \cdot \cdots \cdot \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

Chapter 9: Fermat's Little Theorem

- Definition of reduced residue systems $\pmod{m}$
- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:

(SMT-2019-Discrete-1) How many nonnegative integers less than 2019 are not solutions to $x^8 + 4x^6 - x^2 + 3 \equiv 0 \pmod 7$?

Chapter 10: Euler Theorem

- Definition of the totient function $\phi(n)$
- Using the totient function on basic problems involving relatively prime integers
- Definition of Euler's totient theorem

- Demonstrating that Fermat's little theorem is a special case of Euler's totient theorem

Sample Problem:

(Ata Pir) Find the smallest integer n, such that $\frac{\varphi(n)}{n} < \frac{1}{4}$.

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to more advanced modular arithmetic problems
- Directly computing solutions to systems of congruences
- Using the Chinese remainder theorem backwards

Sample Problem:

(AMC10-2010-A24) The number obtained from the last two nonzero digits of 90! is equal to n. What is n?

(A) 12 **(B)** 32 **(C)** 48 **(D)** 52 **(E)** 68

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Bézout's identity, using the reverse Euclidean Algorithm
- Chicken McNugget theorem
- Finding Pythagorean triples
- Using modular arithmetic to solve Diophantine equations, or to show there is no integer solution

Sample Problem:

(CHMMC-2010 Winter-Individual-9) Let A and B be points in the plane such that $AB = 30$. A circle with integer radius passes through A and B. A point C is constructed on the circle such that $\overline{AC}$ is a diameter of the circle. Compute all possible radii of the circle such that BC is a positive integer.