Topics & Sample Problems

MC35F (AMC 10/12 Advanced Fundamentals)

Contents

Part-I

MC35F Algebra

Chapter 1: Arithmetic

- Word problems using arithmetic with integers, fractions, decimals, and percent
- Decimals with repeating/terminating digits
- Rational/Irrational numbers

Sample Problem:

(BMT-2018-Team-5) How many integers can be expressed in the form:

$$\pm 1 \pm 2 \pm 3 \pm 4 \cdots \pm 2018?$$

Chapter 2: Exponents & Radicals

- Properties of exponents and radicals
- Negative/fractional exponents
- Rationalizing the denominator, simplifying radicals
- Using conjugates of radicals

Sample Problem:

(Lehigh MC-2016-34) What is the smallest integer larger than

$$(\sqrt{5}+\sqrt{3})^6?$$

Note that $3.872 < \sqrt{15} < 3.873$.

Chapter 3: Word Problems & System of Equations

- Word problems, systems of equations in two or more variables

Sample Problem:

(AMC10-2002-B20) Let a, b, and c be real numbers such that

$$a - 7b + 8c = 4 \text{ and } 8a + 4b - c = 7.$$

Then $a^2 - b^2 + c^2$ is

(A) 0 **(B)** 1 **(C)** 4 **(D)** 7 **(E)** 8

Chapter 4: Time, Travel, Work

- Distance = Rate $\times$ Time, average speed
- Harmonic mean
- Relative speed
- Rate/Work Problems

Sample Problem:

(AMC10-2012-A19) Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

Chapter 5: Sequences-1

- Mean, median, mode, range
- Arithmetic and geometric sequences
- Geometric series formula and derivation

Sample Problem:

(AMC10-2000-A23) When the mean, median, and mode of the list $10, 2, 5, 2, 4, 2, x$ are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 17 **(E)** 20

Chapter 6: Sequences-2

- Recurrent sequences
- Finding the general term via patterns

Sample Problem:

(Lehigh MC-2008-33) A Fibonacci-like sequence of numbers is defined by $a_1 = 1$, $a_2 = 3$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. One can compute that $a_{29} = 1149851$ and $a_{30} = 1860498$. What is the value of $\sum_{n=1}^{28} a_n$?

Chapter 7: Functions & Operations

- Definitions of function, domain, codomain/range
- Injective, surjective, bijective functions
- Inverse functions
- Operators
- Simple functional equations

Sample Problem:

(Lehigh MC-2002-36) If $2f(x) + f(1-x) = x^2$ for all x, then $f(x) =$

Chapter 8: Polynomials-1

- Polynomials of a single variable; definitions of degree, root, etc.
- Solving for the roots of a quadratic by factoring, completing the square, or quadratic formula
- Rational root theorem
- Fundamental theorem of algebra
- Less emphasis on complex numbers (Chapter 12)

Sample Problem:

(Aaron Lin, David Zhu) Suppose P is a monic quartic polynomial (i.e. a degree-4 polynomial with leading coefficient 1) such that $P(1) = 1$, $P(2) = 4$, $P(3) = 9$, $P(4) = 16$. Find $P(5)$.

Chapter 9: Polynomials-2

- Generalized Vieta's formulas
- Manipulation of symmetric sums to produce other expressions

Sample Problem:

(HMMT Nov-2016-Guts-27) Let r_1, r_2, r_3, r_4 be the four roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Find the value of

$$\frac{r_1^2}{r_2^2+r_3^2+r_4^2} + \frac{r_2^2}{r_1^2+r_3^2+r_4^2} + \frac{r_3^2}{r_1^2+r_2^2+r_4^2} + \frac{r_4^2}{r_1^2+r_2^2+r_3^2}.$$

Chapter 10: Trigonometry

- Review of trigonometric functions (sin, cos, tan, csc, sec, cot)
- More emphasis on trigonometric identities (addition and multiple-angle formulae)
- Solving algebra problems via trig substitution

Sample Problem:

(BMT-2016-Analysis-5) Find

$$\frac{\tan 1^\circ}{1+\tan 1^\circ} + \frac{\tan 2^\circ}{1+\tan 2^\circ} + \cdots + \frac{\tan 89^\circ}{1+\tan 89^\circ}.$$

Chapter 11: Logarithm

- Definition of a logarithm in base b, simple logarithmic identities (change-of-base formula, addition/subtraction of logarithms)
- Natural logarithms, the number e
- Applications: Binary search, merge sort example

Sample Problem:

(AMC12-2006-B20) Let x be chosen at random from the interval $(0, 1)$. What is the probability that

$$\lfloor \log_{10} 4x \rfloor - \lfloor \log_{10} x \rfloor = 0?$$

Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.

(A) $\frac{1}{8}$ **(B)** $\frac{3}{20}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{1}{4}$

Chapter 12: Complex Numbers

- More rigorous introduction to complex numbers
- Review of the Fundamental Theorem of Algebra, conjugate root theorem
- Polar form, Euler's formula, de Moivre's formula
- Roots of unity

Sample Problem:

(Yitz Deng) Find the number of complex numbers z such that $z^{2015} = \overline{z}$.

MC35F Counting

Chapter 1: Counting Basics

- Addition/multiplication principles
- Permutations, combinations, binomial coefficients

Sample Problem:

(SMT-2018-General-15) How many ways are there to select distinct integers x, y, where $1 \leq x \leq 25$ and $1 \leq y \leq 25$, such that $x + y$ is divisible by 5?

Chapter 2: Casework

- Solving a variety of counting problems using casework
- Use casework to break difficult problems into easier pieces

Sample Problem:

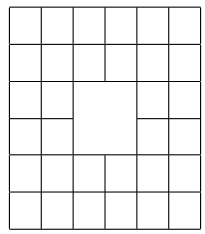
(HMMT Feb-2006-Combinatorics-6) For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

Chapter 3: Complementary Counting & Overcounting

- Solving counting problems using the techniques of complementary counting and/or over-counting

Sample Problem:

(HMMT Feb-2008-Guts-6) Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



Chapter 4: Counting Sets

- Definitions of set, cardinality, union, intersection
- Principle of inclusion-exclusion for two or more sets

Sample Problem:

(HMMT Feb-2010-Combinatorics-1) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

Chapter 5: Counting with Digits

- Solving a variety of counting problems involving digits of a number
- Counting palindromes

Sample Problem:

(AMC12-2008-A21) A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ is heavy-tailed if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

(A) 36 **(B)** 40 **(C)** 44 **(D)** 48 **(E)** 52

Chapter 6: Path Counting & Bijections

- Definitions of injective, surjective, and bijective functions
- Examples of bijections between two infinite sets (e.g. the set of whole numbers and the set of integers)
- Solving counting problems by establishing a bijection

Sample Problem:

(AMC12-2010-A18) A 16-step path is to go from $(-4,-4)$ to $(4,4)$ with each step increasing either the x-coordinate or y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2, -2 \le y \le 2$ at each step?

(A) 92 **(B)** 144 **(C)** 1568 **(D)** 1698 **(E)** 12,800

Chapter 7: Stars and Bars

- Using the stars and bars technique to solve a variety of counting problems

Sample Problem:

(Caleb Ji) How many ways can David pick four of the first twelve positive integers such that no two of the numbers he picks are consecutive?

Chapter 8: Binomial

- Binomial theorem, Pascal's triangle, Sierpinski's triangle
- Various combinatorial identities, such as the hockey stick identity

Sample Problem:

(AMC10-2011-B23) What is the hundreds digit of 2011^{2011}?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

Chapter 9: Counting with Recursion

- Solving counting problems by setting up a recursion and/or finding patterns

Sample Problem:

(Lehigh MC-2014-26) How many 10-digit strings of 0?s and 1?s are there that do not contain any consecutive 0?s?

Chapter 10: Probability-1

- Basic probability definitions and axioms
- Definitions of complementary events, independence, disjoint events

Sample Problem:

(AMC10-2004-B23) Each face of a cube is painted either red or blue, each with probability $\frac{1}{2}$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

Chapter 11: Probability-2

- Conditional probability, Bayes' theorem
- Geometric probability

Sample Problem:

(AMC10-2012-A25) Real numbers x, y, and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Chapter 12: Expected Value

- Expected value and linearity of expectation (for an arbitrary number of events)
- Introduction to state diagrams, Markov chains

Sample Problem:

(Bill Huang) 10 boys and 10 girls sit in a row. Let x be the number of adjacent boy-girl (or girl-boy) pairs and y be the number of adjacent girl-girl pairs. What is the expected value of $x - y$? Express your answer as a common fraction in reduced form.