Topics & Sample Problems

MC30F (AMC 10/12 Basic Fundamentals)

alphastar academy

Contents

Part-II

MC30F Geometry

Chapter 1: Angles

- Angles (review)
- Inscribed angles in a circle, cyclic quadrilaterals

Sample Problem:

(Alec Sun) In regular nonagon (9-sided polygon) $ABCDEFGHI$, draw the circle that is tangent to IA at A and CD at C. What is the degree measure of minor arc AC?

Chapter 2: Special Triangles

- 30-60-90, 45-45-90, and 15-75-90 triangles
- Pythagorean triples

Sample Problem:

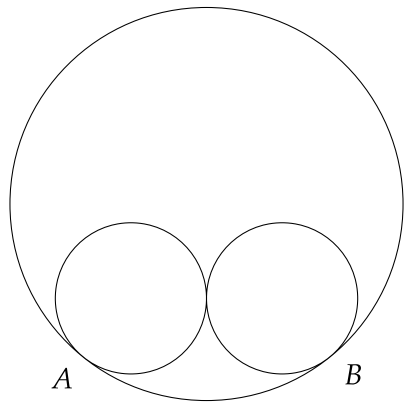
(Math Day at the Beach-2012-Team-1) Each of two congruent equilateral triangles with side s has center that is a vertex of the other triangle. What is the area of the overlap, in terms of s?

Chapter 3: Similarity

- Similarity/congruence axioms (SSS, SAS, ASA, AA similarity)
- Power of a point, angle bisector theorem

Sample Problem:

(AMC10-2018-A15) Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



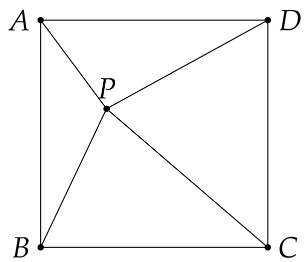
(A) 21 **(B)** 29 **(C)** 58 **(D)** 69 **(E)** 93

Chapter 4: Special Points

- Special points of a triangle (centroid, incenter, circumcenter, orthocenter)

Sample Problem:

(AMC12-2018-B13) Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?



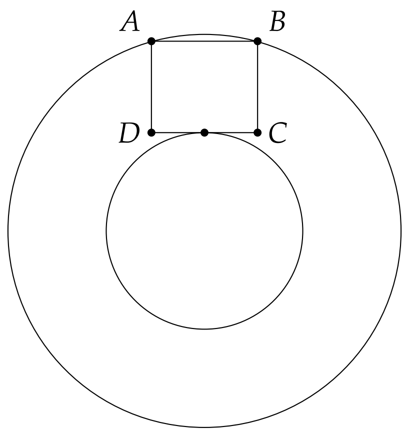
Chapter 5: Length-1

- Triangle inequality, Ravi substitution

- Pythagorean theorem, distance formula
- Stewart's theorem

Sample Problem:

(AMC12-2012-A12) A square region $ABCD$ is externally tangent to the circle with equation $x^2 + y^2 = 1$ at the point $(0, 1)$ on the side CD. Vertices A and B are on the circle with equation $x^2 + y^2 = 4$. What is the side length of the square?



(A) $\frac{\sqrt{10}+5}{10}$ **(B)** $\frac{2\sqrt{5}}{5}$ **(C)** $\frac{2\sqrt{2}}{3}$ **(D)** $\frac{2\sqrt{19}-4}{5}$ **(E)** $\frac{9-\sqrt{17}}{5}$

Chapter 6: Length-2

- Mass points considering levers/torque
- Ceva's theorem, Menelaus' theorem

Sample Problem:

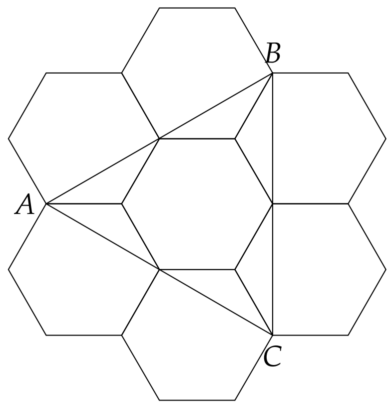
(Lehigh MC-2014-27) Let BE be a median of triangle ABC, and let D be a point on AB such that $BD/DA = 3/7$. What is the ratio of the area of triangle BED to that of triangle ABC? Express your answer as a common fraction in reduced form.

Chapter 7: Area-1

- Areas of simple shapes (triangle, certain quadrilaterals)
- Triangle area formulas (Heron's formula, A = rs, A = abc/4R, (ab sin C)/2

Sample Problem:

(AMC10-2014-B13) Six regular hexagons surround a regular hexagon of side length 1 as shown. What is the area of $\triangle ABC$?



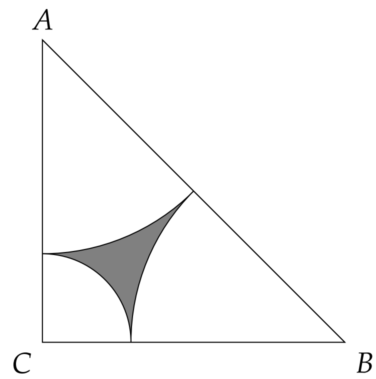
(A) $2\sqrt{3}$ **(B)** $3\sqrt{3}$ **(C)** $1+3\sqrt{2}$ **(D)** $2+2\sqrt{3}$ **(E)** $3+2\sqrt{3}$

Chapter 8: Area-2

- Area formula for a circle, sector
- Brahmagupta's formula
- Area of more complicated shapes involving circles and/or other polygons

Sample Problem:

(ARML-0000-Team-1) In ΔABC, $m\angle A = m\angle B = 45^\circ$ and $AB = 16$. Mutually tangent circular arcs are drawn centered at all three vertices; the arcs centered at A and B intersect at the midpoint of $\overline{AB}$. Compute the area of the region inside the triangle and outside of the three arcs.

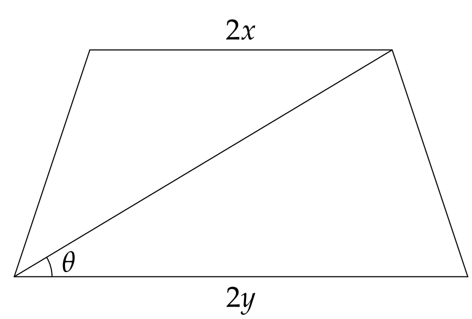


Chapter 9: Trigonometry-1

- Definitions of sin, cos, tan, as well as csc, sec, cot
- The unit circle
- Basic trig identities, Sum and difference formulas for sin, cos, tan (e.g. sin(a+b))

Sample Problem:

(UK MC-2010-Senior-14) The parallel sides of a trapezium (trapezoid) have lengths $2x$ and $2y$, respectively. The diagonals are equal in length, and a diagonal makes an angle θ with the parallel sides, as shown. What is the length of each diagonal?



(A) $x + y$ **(B)** $\frac{x+y}{\sin\theta}$ **(C)** $(x + y)\cos\theta$ **(D)** $(x + y)\tan\theta$ **(E)** $\frac{x+y}{\cos\theta}$

Chapter 10: Trigonometry-2

- Law of sines
- Law of cosines
- Ratio lemma, trig Ceva's theorem
- Solving algebra problems by trig substitution

Sample Problem:

(Hong Kong MC-2006-17) If a square can completely cover a triangle with side lengths 3, 4 and 5, find the smallest possible side length of the square.

Chapter 11: Analytic Geometry

- Slope, equation of a line using slope-intercept or point-slope form, distance and midpoint formulas
- Reflections over lines in the coordinate plan
- Equation of a circle
- Shoelace formula

Sample Problem:

(AMC12-2006-B16) Regular hexagon $ABCDEF$ has vertices A and C at $(0, 0)$ and $(7, 1)$, respectively. What is its area?

(A) $20\sqrt{3}$ **(B)** $22\sqrt{3}$ **(C)** $25\sqrt{3}$ **(D)** $27\sqrt{3}$ **(E)** 50

Chapter 12: 3D

- Distance formula in 3D
- Area/volume of various 3D shapes (cube, prisms, cylinders, cones, spheres)
- Common 3D solids
- Euler's polyhedral formula

Sample Problem:

(HMMT Feb-2004-Guts-18) On a spherical planet with diameter 10,000 km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within 5,000 km of ground zero and leave anything beyond 5,000 km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

MC30F Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences
- Sum of squares and sum of cubes formula
- Sigma notation

Sample Problem:

(Australian MC-2000-I22) The number 2000 is expressed as the sum of 32 consecutive positive integers. The largest of these integers is

(A) 33 **(B)** 42 **(C)** 77 **(D)** 78 **(E)** 79

Chapter 2: Primes & Prime Factorization

- Definition of divisibility ($a \mid b$)
- Prime and composite numbers, Euclid's proof of the infinitude of primes
- Fundamental theorem of arithmetic
- Legendre's formula
- Prime number theorem, prime checking algorithms (optional)

Sample Problem:

(UK MC-2008-Senior-19) How many prime numbers p are there such that $199p + 1$ is a perfect square?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

Chapter 3: Divisibility

- Divisibility by numbers 2-11 inclusive
- Various problems involving divisibility, prime factorization, etc.

Sample Problem:

(Richard Spence) Find the largest positive integer multiple of 11 whose digits are all odd and different.

Chapter 4: Number & Sum of Divisors

- General formula for the number and sum of divisors of a positive integer n, given its prime factorization
- Perfect, abundant, and deficient numbers (optional)

Sample Problem:

(Lehigh MC-2006-32) Let S denote the set of all (positive) divisors of 60^5. The product of all the numbers in S equals 60^e for some integer e. What is the value of e?

Chapter 5: Factoring Techniques

- Difference of squares
- Sum and difference of cubes; sum and difference of n-th powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity
- Informal definition of an irreducible polynomial over the integers (e.g. $x^2 + y^2$)

Sample Problem:

(Lehigh MC-2004-37) How many ordered pairs (x, y) of integers satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$? (Note that both positive and negative integers are allowed.)

Chapter 6: Number Bases

- Conversion between different number bases (emphasis on base 2, 8, 10, and 16)
- Arithmetic in different bases

- Fast base conversion (e.g. binary to hexadecimal)

Sample Problem:

(AMC12-2012-B11) In the equation below, A and B are consecutive positive integers, and A, B, and $A+B$ represent number bases:

$$132_A + 43_B = 69_{A+B}$$

What is $A+B$?

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

Chapter 7: GCD & LCM

- Definition of relatively prime
- Computing the GCD and LCM using the prime factorization
- Computing the GCD of two numbers using the Euclidean algorithm

Sample Problem:

(BMT-2013-Individual-5) Two positive integers m and n satisfy

$$\max(m,n) = (m-n)^2$$
$$\gcd(m,n) = \frac{\min(m,n)}{6}$$

Find $\operatorname{lcm}(m,n)$.

Chapter 8: Modular Arithmetic

- Basic properties of the modulo (reflexive, symmetric, transitive, etc.)
- Proof of divisibility rules using modular arithmetic
- Modular inverses
- More advanced modulo calculations involving basic operations

Sample Problem:

(Metehan Ozsoy) What is the remainder when $(1+2+\cdots+49)^{49}$ is divided by 50?

Chapter 9: Fermat's Little Theorem

- Definition of reduced residue systems $\pmod{m}$
- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:

(BMT-2019-Team-2) Find the remainder when 2^{2019} is divided by 7.

Chapter 10: Euler Theorem

- Definition of the totient function $\phi(n)$
- Using the totient function on basic problems involving relatively prime integers
- Definition of Euler's totient theorem
- Demonstrating that Fermat's little theorem is a special case of Euler's totient theorem

Sample Problem:

(Richard Spence) What is the sum of all positive integers less than 720 which are relatively prime to 720?

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to more advanced modular arithmetic problems
- Directly computing solutions to systems of congruences
- Using the Chinese remainder theorem backwards

Sample Problem:

(Richard Spence) Find all 3-digit positive integers N such that the numbers N, $N+1$, and $N+2$ are divisible by 7, 8, and 9 respectively.

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Bézout's identity, using the reverse Euclidean Algorithm
- Chicken McNugget theorem
- Finding Pythagorean triples

- Using modular arithmetic to solve Diophantine equations, or to show there is no integer solution

Sample Problem:

(AIME-2012-II-1) Find the number of ordered pairs of positive integer solutions (m, n) to the equation $20m + 12n = 2012$.