Topics & Sample Problems

MC30F (AMC 10/12 Basic Fundamentals)

alphastar academy

Contents

Part-I

MC30F Algebra

Chapter 1: Arithmetic

- Word problems using arithmetic with integers, fractions, decimals, and percent
- Decimals with repeating/terminating digits
- Rational/Irrational numbers

Sample Problem:

(AMC10-2009-A5) What is the sum of the digits of the square of $111,111,111$?

(A) 18 **(B)** 27 **(C)** 45 **(D)** 63 **(E)** 81

Chapter 2: Exponents & Radicals

- Properties of exponents and radicals
- Negative/fractional exponents
- Rationalizing the denominator, simplifying radicals
- Using conjugates of radicals

Sample Problem:

(Lehigh MC-2008-10) Simplify $\sqrt{19+\sqrt{297}} - \sqrt{19-\sqrt{297}}$.

Chapter 3: Word Problems

- Word problems, systems of equations in two or more variables

Sample Problem:

(SMT-2012-General-4) Steve works 40 hours a week at his new job. He usually gets paid 8 dollars an hour, but if he works for more than 8 hours on a given day, he earns 12 dollars an hour for every additional hour over 8 hours. If x is the maximum number of dollars that Steve can earn in one week by working exactly 40 hours, and y is the minimum number of dollars that Steve can earn in one week by working exactly 40 hours, what is $x - y$?

Chapter 4: Distance, Rate, and Time

- Distance = Rate × Time, average speed
- Harmonic mean
- Relative speed
- Rate/Work Problems

Sample Problem:

(AMC10-2002-A12) Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

(A) 45 **(B)** 48 **(C)** 50 **(D)** 55 **(E)** 58

Chapter 5: Sequences-1

- Mean, median, mode, range
- Arithmetic and geometric sequences
- Geometric series formula and derivation

Sample Problem:

(AMC10-2010-B17) Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Chapter 6: Sequences-2

- Recurrent sequences
- Finding the general term via patterns

Sample Problem:

(Math Day at the Beach-2018-Individual-14) Form the sequence such that $x_1 = x_2 = 1$, and for $n > 2$, $x_n = x_{n-1}^2 + x_{n-2}$. Of the numbers $x_1,\ x_2, \ldots, x_{2018}$, how many are divisible by 3?

Chapter 7: Functions & Operations

- Definitions of function, domain, codomain/range
- Injective, surjective, bijective functions
- Inverse functions
- Operators
- Simple functional equations

Sample Problem:

(Math Day at the Beach-2014-Individual-18) Compute

$$\sum_{n=1}^{99} \lfloor 0.67n \rfloor,$$

where the notation $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.

Chapter 8: Polynomials-1

- Polynomials of a single variable; definitions of degree, root, etc.
- Solving for the roots of a quadratic by factoring, completing the square, or quadratic formula
- Rational root theorem
- Fundamental theorem of algebra
- Less emphasis on complex numbers (Chapter 12)

Sample Problem:

(HMMT Nov-2012-Guts-15) Find the area of the region in the xy-plane consisting of all points (a, b) such that the quadratic

$$ax^2 + 2(a + b - 7)x + 2b = 0$$

has fewer than two real solutions for x.

Chapter 9: Polynomials-2

- Generalized Vieta's formulas
- Manipulation of symmetric sums to produce other expressions

Sample Problem:

(Justin Stevens) If r, s, and t are the roots of $f(x) = 3x^3 - 9x^2 + 3x - 7$, what is the value of $(3 - r)(3 - s)(3 - t)$? Express your answer as a common fraction in reduced form.

Chapter 10: Trigonometry

- Review of trigonometric functions (sin, cos, tan, csc, sec, cot)
- More emphasis on trigonometric identities (addition and multiple-angle formulae)
- Solving algebra problems via trig substitution

Sample Problem:

(Richard Spence) How many real numbers $\theta \in [0, 2\pi)$ (in radians) are there such that $\sin\theta = \sin 6\theta$?

Chapter 11: Logarithms

- Definition of a logarithm in base b, simple logarithmic identities (change-of-base formula, addition/subtraction of logarithms)
- Natural logarithms, the number e
- Applications: Binary search, merge sort example

Sample Problem:

(AMC12-2018-A14) The solution to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 5 **(B)** 13 **(C)** 17 **(D)** 31 **(E)** 35

Chapter 12: Complex Numbers

- More rigorous introduction to complex numbers
- Review of the Fundamental Theorem of Algebra, conjugate root theorem
- Polar form, Euler's formula, de Moivre's formula
- Roots of unity

Sample Problem:

(BMT-2016-Individual-5) Positive integers x, y, z satisfy $(x + yi)^2 - 46i = z$. What is $x + y + z$?

MC30F Counting

Chapter 1: Counting Basics

- Addition/multiplication principles
- Permutations, combinations, binomial coefficients

Sample Problem:

(AMC10-2006-A18) A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

(A) $10^4 \cdot 26^2$ **(B)** $10^3 \cdot 26^3$ **(C)** $5 \cdot 10^4 \cdot 26^2$ **(D)** $10^2 \cdot 26^4$ **(E)** $5 \cdot 10^3 \cdot 26^3$

Chapter 2: Casework

- Solving a variety of counting problems using casework
- Use casework to break difficult problems into easier pieces

Sample Problem:

(AMC12-2014-A13) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

(A) 2100 **(B)** 2220 **(C)** 3000 **(D)** 3120 **(E)** 3125

Chapter 3: Complementary Counting & Overcounting

- Solving counting problems using the techniques of complementary counting and/or overcounting

Sample Problem:

(AMC12-2012-B12) How many sequences of zeros and/or ones of length 20 have all zeros consecutive, or all the ones consecutive, or both?

(A) 190 **(B)** 192 **(C)** 211 **(D)** 380 **(E)** 382

Chapter 4: Counting Sets

- Definitions of set, cardinality, union, intersection
- Principle of inclusion-exclusion for two or more sets

Sample Problem:

(Justin Stevens) Out of the 140 seniors, 70 play basketball, 100 play soccer, and 30 play hockey. 44 seniors play soccer and basketball, 12 play basketball and hockey, and 9 play soccer and hockey. How many seniors play all three of these sports?

Chapter 5: Counting with Digits

- Solving a variety of counting problems involving digits of a number
- Counting palindromes

Sample Problem:

(Jamie Gu) How many 5-digit positive integers, such as 53550, contain exactly three 5's?

Chapter 6: Path Counting & Bijections

- Definitions of injective, surjective, and bijective functions
- Examples of bijections between two infinite sets (e.g. the set of whole numbers and the set of integers)
- Solving counting problems by establishing a bijection

Sample Problem:

(Evan Chen) Determine the number of sequences of positive integers

$$1 = x_0 < x_1 < \ldots < x_{10} = 10^5$$

with the property that for each $m = 0, \ldots, 9$ the number $\dfrac{x_{m+1}}{x_m}$ is a prime number.

Chapter 7: Stars and Bars

- Using the stars and bars technique to solve a variety of counting problems

Sample Problem:

(HMMT Feb-2010-Guts-12) How many different collections of 9 letters are there? A letter can appear multiple times in a collection. Two collections are equal if each letter appears the same number of times in both collections.

Chapter 8: Binomial

- Binomial theorem, Pascal's triangle, Sierpinski's triangle
- Various combinatorial identities, such as the hockey stick identity

Sample Problem:

(AlphaStar) What is the coefficient of x^{42} in the expansion of

$$(1+x+x^2+\cdots+x^{39})(1+x+\cdots+x^{40})(1+x+\cdots+x^{41})?$$

Chapter 9: Counting with Recursion

- Solving counting problems by setting up a recursion and/or finding patterns

Sample Problem:

(Ali Gurel) How many ways can a 2×12 rectangle be tiled using twelve non-overlapping 1×2 tiles? Tiles may be positioned vertically or horizontally.

Chapter 10: Probability-1

- Basic probability definitions and axioms
- Definitions of complementary events, independence, disjoint events

Sample Problem:

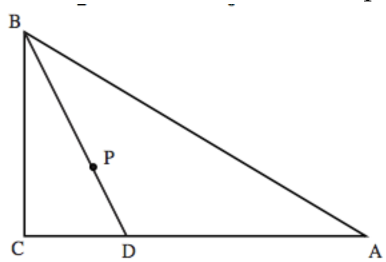
(AIME-2002-I-1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Chapter 11: Probability-2

- Conditional probability, Bayes' theorem
- Geometric probability

Sample Problem:

(AMC12-2002-A22) Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $m\angle ABC = 60°$, and $AB = 10$. Let P be randomly chosen inside $\triangle ABC$, and extend $\overline{BP}$ to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?



(A) $\frac{2-\sqrt{2}}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{3}}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5-\sqrt{5}}{5}$

Chapter 12: Expected Value

- Expected value and linearity of expectation (for an arbitrary number of events)
- Introduction to state diagrams, Markov chains

Sample Problem:

(AMC12-2016-B19) Tom, Dick, and Harry are playing a game. Starting at the same time, each of them flips a fair coin repeatedly until he gets his first head, at which point he stops. What is the probability that all three flip their coins the same number of times?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{7}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$