Topics & Sample Problems

MC25F (AMC 8/MathCounts Advanced Fundamentals)

alphastar academy

Contents

Part-II

MC25F Geometry

Chapter 1: Angles

- Definitions of acute, right, obtuse, complementary, and supplementary angles
- Parallel, perpendicular, and transversal lines
- Sum of the degree measures in a triangle, different types of triangles
- Inscribed angles and arcs in circles

Sample Problem:

(Sean Shi) In triangle ABC, $m\angle A = 42^\circ$. The angle bisectors of $\angle B$ and $\angle C$ intersect at I. What is $m\angle BIC$ in degrees?

Chapter 2: Pythagorean Theorem, Special Triangles

- 30-60-90 and 45-45-90 triangles
- Pythagorean theorem and Pythagorean triples

Sample Problem:

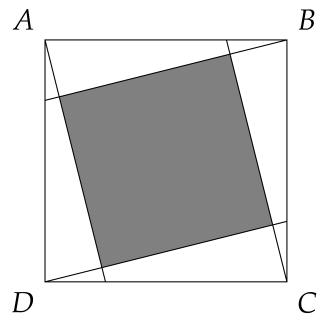
(Wanlin Li) Convex quadrilateral $ABCD$ has $m\angle B = m\angle D = 90^\circ$. If $AC = 2\sqrt{6}$, $AB = \sqrt{6}$, and $AD = 2\sqrt{3}$, what is the area of quadrilateral $ABCD$?

Chapter 3: Similarity

- Congruence and similarity axioms (SSS, SAS, ASA, AA)
- SSA is not a congruence axiom
- Angle bisector theorem

Sample Problem:

(BmMT-2012-Team-8) As pictured, lines are drawn from the vertices of a unit square to an opposite trisection point. If each triangle has legs with ratio 3:1, what is the area of the shaded region? Express your answer as a common fraction in reduced form.

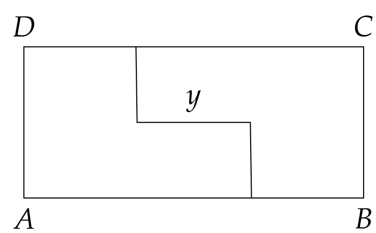


Chapter 4: Length-1

- Perimeter of polygons
- Triangle inequality
- Review of the Pythagorean theorem

Sample Problem:

(AMC10-2006-A7) The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?



(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

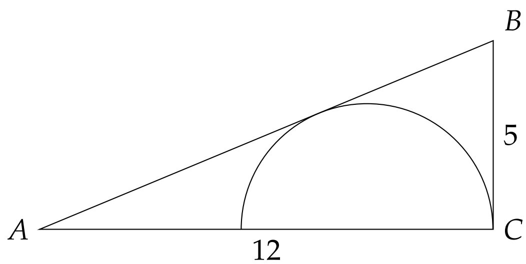
Chapter 5: Length-2

- Circumference of a circle

- Power of a point
- Inscribed and circumscribed circles of a triangle
- Ravi substitution

Sample Problem:

(AMC8-2017-22) In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



(A) $\frac{7}{6}$ **(B)** $\frac{13}{5}$ **(C)** $\frac{59}{18}$ **(D)** $\frac{10}{3}$ **(E)** $\frac{60}{13}$

Chapter 6: Length-3

- Introduction to the mass points technique using physics concepts (levers, torque)
- Ceva's theorem and Menelaus' theorem

Sample Problem:

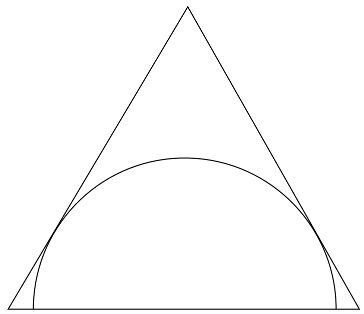
(Ali Gurel) In $\triangle ABC$, points D, E, and F are on BC, CA, and AB, respectively. Cevians AD, BE, and CF intersect at P. If $\dfrac{AF}{AB} = \dfrac{1}{3}$ and $\dfrac{AE}{AC} = \dfrac{1}{4}$, what is $\dfrac{AP}{AD}$? Express your answer as a common fraction in reduced form.

Chapter 7: Area-1

- Unit conversions (e.g. square feet to square yards)
- Areas of simple polygons (squares, rectangles, triangles, trapezoids)
- Other formulas for the area of a triangle, including Heron's

Sample Problem:

(AMC8-2016-25) A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?



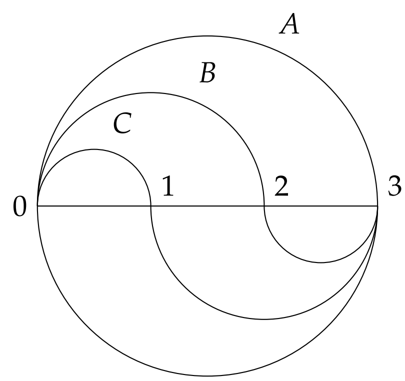
(A) $4\sqrt{3}$ **(B)** $\frac{120}{17}$ **(C)** 10 **(D)** $\frac{17\sqrt{2}}{2}$ **(E)** $\frac{17\sqrt{3}}{2}$

Chapter 8: Area-2

- Area of a circle and sector

Sample Problem:

(UNB-2008-Gr 9-24) A is a circle whose diameter is equal to 3 units. Curves B and C are respectively made from one half-circle of diameter equal to 1 unit and one half-circle of diameter equal to 2 units. What is the area of the region located between curves B and C?



(A) $\frac{3}{4}$ **(B)** $\frac{3\pi}{4}$ **(C)** 3 **(D)** 3π **(E)** None of these

Chapter 9: Analytic Geometry-1

- Cartesian coordinate system (2 dimensions)
- Slope-intercept and point-slope form of a line
- Midpoint and distance formula
- Solving geometry problems by using coordinates

Sample Problem:

(AMC12-2018-B3) A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these two lines?

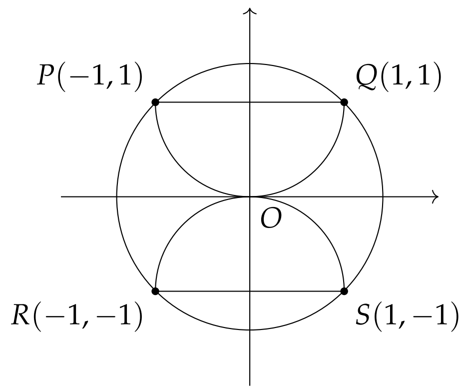
(A) 5 **(B)** 10 **(C)** 20 **(D)** 25 **(E)** 50

Chapter 10: Analytic Geometry-2

- Reflecting/rotating a point in the coordinate plane
- General equation of a circle in the coordinate plane
- Area of a polygon with Shoelace formula

Sample Problem:

(AMC8-2010-23) Semicircles POQ and ROS pass through the center of circle O. What is the ratio of the combined areas of the two semicircles to the area of circle O?



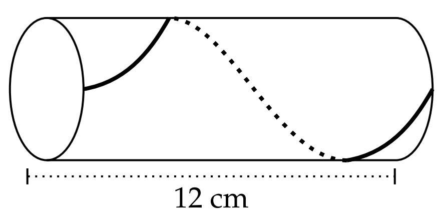
(A) $\frac{\sqrt{2}}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{\pi}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{\sqrt{2}}{2}$

Chapter 11: 3D-1

- Applications of 3D geometry in the real world
- Applying 2D geometry techniques to 3D space, 3D distance formula
- Surface area of various polyhedra, cylinders, cones, spheres

Sample Problem:

(Alexander Parr) A paper towel roll has a base circumference of 5 cm and a height of 12 cm. A crease (shown in bold) wraps around the paper towel roll once. What is the length of the crease, in cm?



Chapter 12: 3D-2

- Volume of various 3D shapes (polyhedra, cylinders, cones, spheres)
- Volume of more complex shapes

Sample Problem:

(Alexander Parr) A cube is inscribed inside of a sphere with a radius of 1 cm. What is the volume of the region inside the sphere but outside the cube?

MC25F Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences (e.g. sum of the first n positive integers)
- Sum of the first n perfect squares, cubes

Sample Problem:

(Richard Spence) The expression $1^3 + 2^3 + 3^3 + \ldots + n^3$ is equal to a perfect fourth power, where $n > 1$. What is the smallest possible value of n?

Chapter 2: Primes & Prime Factorization

- Definition of divisibility
- Fundamental Theorem of Arithmetic
- Determining if a number is prime or not
- Legendre's formula

Sample Problem:

(AMC10-2002-B6) For how many positive integers n is $n^2 - 3n + 2$ a prime number?

(A) none **(B)** one **(C)** two **(D)** more than two, but finitely many
(E) infinitely many

Chapter 3: Divisibility

- Divisibility rules for all positive integers up to and including 11

Sample Problem:

(Jennifer Zhang) The 7-digit number $\underline{2014ABC}$, where A, B, and C each represent digits, is divisible by 2, 3, 5, and 11. What is the greatest possible value of $\underline{2014ABC}$?

Chapter 4: Number of Divisors

- Determining the number of divisors of a positive integer n using the prime factorization of n
- Multiplicative functions

Sample Problem:

(Tiancheng Qin) Given that b and n are both positive integers at most 15, what is the greatest number of divisors that b^n can have?

Chapter 5: Sum of Divisors

- Definition of $\sigma(n)$
- Determining the sum of divisors of a number n using the prime factorization of n

Sample Problem:

(Ali Gurel) Given that $\sigma(123) = 168$, what is $\sigma(246)$?

Chapter 6: Factoring Techniques

- Difference of squares
- Simon's Favorite Factoring Trick (SFFT)
- Sum of cubes, difference of cubes
- Sophie-Germain identity

Sample Problem:

(Ali Gurel) Find the sum of prime divisors of 4891.

Chapter 7: Number Bases

- Representing numbers in different bases
- Converting numbers between bases (emphasis on base 2, 8, and 16)
- Arithmetic in different bases

Sample Problem:

(Nathan Zhang) What is the base 10 value of the sum $11_2 + 22_3 + 33_4 + \ldots + 99_{10}$?

Chapter 8: GCD & LCM

- Computing the GCD and LCM of two or more numbers
- Euclidean algorithm
- Relation between gcd and lcm ($lcm(a, b) = ab/gcd(a, b)$)

Sample Problem:

(Ali Gurel) How many ordered pairs of positive integers (a, b) are there such that $\text{lcm}(a, b) = 48$ and $\text{gcd}(a, b) = 4$?

Chapter 9: Modular Arithmetic

- Introduction to the congruence operator ($a \equiv b \pmod{m}$)
- Basic properties of modulo (reflexive, symmetric, transitive)
- Computing remainders by finding patterns
- Proof of the divisibility rules for 3, 9, and 11

Sample Problem:

(Kevin Chang) The Fibonacci sequence is the sequence $1, 1, 2, 3, 5, \ldots$, where each term after the second is the sum of the previous two terms. What is the units digit of the 10^6th Fibonacci number?

Chapter 10: FLT and Euler's Totient Theorem

- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:

(Richard Spence) What is the units digit of $1^{12} + 2^{12} + 3^{12} + \ldots + 2019^{12}$?

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to basic modular arithmetic problems
- Solving basic systems of congruences

Sample Problem:

(Kevin Chang) How many integers between 1 and 1260 inclusive are divisible by 36, but not by 5 or 7?

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Chicken McNugget theorem
- Bézout's identity
- Pythagorean triples
- Using modular arithmetic to show that a Diophantine equation has no solutions

Sample Problem:

(Richard Spence) Hexagonal-shaped tubing is sold in packages of 7 and 19 tubes. What is the smallest number k such that for any $n \geq k$, I can always buy exactly n tubes?