# MC45N
# AIME Advanced Number Theory

## Chapter 1: Number Bases

- Non-decimal bases
- Legendre's formula

**Sample Problem:**
(AIME-2010-I-10) Let $N$ be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the $a_i$'s are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find $N$.

## Chapter 2: Primes & Prime Factorization

- Definitions of primes and Euclid's Lemma
- Fundamental Theorem of Arithmetic

**Sample Problem:** (1001 Problems in NT p26 q153) Find the smallest positive integer $n$ such that $n/2$ is a perfect square, $n/3$ is a cube and $n/5$ is a fifth power.

## Chapter 3: Divisibility Rules

- Divisibility rules

- p-adic valuation
- Lifting the exponent

**Sample Problem:**
(AIME-2006-II-14) Let $S_n$ be the sum of the reciprocals of the nonzero digits of the integers from 1 to $10^n$, inclusive. Find the smallest positive integer $n$ for which $S_n$ is an integer.

## Chapter 4: Multiplicative Functions

- Problems involving multiplicative functions, such as Divisor function, Sigma function, Totient function
- Properties of $\varphi$ function

**Sample Problem:**
(AIME-2016-II-11) For positive integers $N$ and $k$, define $N$ to be $k$-nice if there exists a positive integer $a$ such that $a^k$ has exactly $N$ positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

## Chapter 5: Factoring Techniques

- Difference of squares and arbitrary powers, sum of cubes and odd powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity

**Sample Problem:**
(CHMMC-2012 Fall-Individual-8) Find two pairs of positive integers $(a, b)$ with $a > b$ such that

$$a^2 + b^2 = 40501.$$

## Chapter 6: GCD & LCM

- Greatest common divisor, least common multiple
- Euclidean algorithm and its applications

- Bezout's identity

**Sample Problem:**
(CHMMC-2010 Fall-Team-5) The three positive integers $a, b, c$ satisfy the equalities $\gcd(ab, c^2) = 20$, $\gcd(ac, b^2) = 18$, and $\gcd(bc, a^2) = 75$. Compute the minimum possible value of $a + b + c$.

# Chapter 7: Modular Arithmetic

- Properties of modulo
- Modular inverses
- Using binomial theorem to find remainders
- Solving AIME level problems using modular arithmetic

**Sample Problem:**
(AIME-2012-I-15) There are $n$ mathematicians seated around a circular table with $n$ seats numbered 1, 2, 3, ..., $n$ in clockwise order. After a break they again sit around the table. The mathematicians note that there is a positive integer $a$ such that

1. for each $k$, the mathematician who was seated in seat $k$ before the break is seated in seat $ka$ after the break (where seat $i + n$ is seat $i$);
2. for every pair of mathematicians, the number of mathematicians sitting between them after the break, counting in both the clockwise and the counter-clockwise directions, is different from either of the number of mathematicians sitting between them before the break.

Find the number of possible values of $n$ with $1 < n < 1000$.

# Chapter 8: Fermat's Little Theorem & Euler Theorem

- Fermat's little theorem
- Euler's totient theorem
- Wilson's theorem

**Sample Problem:**
(HMMT Feb-2010-Guts-29) Compute the remainder when

$$\sum_{k=1}^{30303} k^k$$

is divided by 101.

# Chapter 9: Chinese Remainder Theorem

- Chinese remainder theorem (CRT)
- Computing solutions to CRT, using CRT backwards

**Sample Problem:**
(AIME-2012-I-10) Let $\mathcal{S}$ be the set of all perfect squares whose rightmost three digits in base 10 are 256. Let $\mathcal{T}$ be the set of all numbers of the form $\frac{x-256}{1000}$, where $x$ is in $\mathcal{S}$. In other words, $\mathcal{T}$ is the set of numbers that result when the last three digits of each number in $\mathcal{S}$ are truncated. Find the remainder when the tenth smallest element of $\mathcal{T}$ is divided by 1000.

# Chapter 10: Degree

- Definition and properties of order modulo m

**Sample Problem:**
(AIME-2018-I-11) Find the least positive integer $n$ such that when $3^n$ is written in base 143, its two right-most-digits in base 143 are 01.

# Chapter 11: Primitive Roots

- Definition of primitive roots
- Primitive root theorem
- Finding number of equations of modular equations using primitive roots

**Sample Problem:**
(CHMMC-2010 Winter-Individual-15) Compute the number of primes $p$ less than 100 such that $p$ divides $n^2 + n + 1$ for some integer $n$.

# Chapter 12: Diophantine Equations

- General strategies for solving AIME level diophantine equations
- Chicken McNugget Theorem
- Computing Pythagorean triples
- Fermat's Last Theorem (optional)

**Sample Problem:**
(AIME-2008-II-15) Find the largest integer $n$ satisfying the following conditions:
(i) $n^2$ can be expressed as the difference of two consecutive cubes;
(ii) $2n + 79$ is a perfect square.