MC45C
AIME Advanced Counting

Chapter 1: Basic Counting Techniques

- Solving counting problems using techniques such as casework and complementary counting

Sample Problem:
(AIME-2016-I-8) For a permutation $p = (a_1, a_2, \ldots, a_9)$ of the digits $1, 2, \ldots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find $|m - n|$.

Chapter 2: Counting Sets & PIE

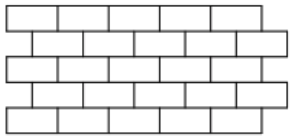
- Solving counting problems using the Principle of Inclusion and Exclusion (PIE)

Sample Problem:
(HMMT Feb-2006-Combinatorics-1) Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

Chapter 3: Path Counting & Bijections

- Solving counting problems using bijections
- Solving path-counting problems

Sample Problem:
(HMMT Feb-2012-Combinatorics-3) In the figure below, how many ways are there to select 5 bricks, one in each row, such that any two bricks in adjacent rows are adjacent?



Chapter 4: Stars and Bars

- Solving counting problems using the Stars and Bars method

Sample Problem:
(PUMaC-2014-Team-5) How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a + b + c = 103$?

Chapter 5: Binomial

- Solving counting problems involving binomials and multinomials
- Binomial identities such as Hockey-Stick Identity and Vandermonde's Identity

Sample Problem:
(PUMaC-2008-Combinatorics-4) Find the sum of the values of x for which

$$\binom{x}{0} - \binom{x}{1} + \binom{x}{2} - \cdots + \binom{x}{2008} = 0.$$

Chapter 6: Counting with Recursion

- Identifying which counting problems can be solved using recursions
- Finding and solving recursions

Sample Problem:
(HMMT Feb-2002-Guts-13) A *domino* is a 1-by-2 or 2-by-1 rectangle. A *domino tiling* of a region of the plane is a way of covering it (and only it) completely by non-overlapping dominoes. For instance, there is one domino tiling of a 2-by-1 rectangle and there are 2 tilings of a 2-by-2 rectangle (one consisting of two horizontal dominoes and one consisting of two vertical dominoes). How many domino tilings are there of a 2-by-10 rectangle?

Chapter 7: Probability

- Solving difficult probability problems
- Conditional probability and Bayes' Theorem
- Geometric probability

Sample Problem:
(HMMT Feb-2010-Guts-16) Jessica has three marbles colored red, green, and blue. She randomly selects a non-empty subset of them (such that each subset is equally likely) and puts them in a bag. You then draw three marbles from the bag with replacement. The colors you see are red, blue, red. What is the probability that the only marbles in the bag are red and blue?

Chapter 8: Expected Value

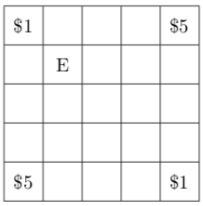
- Random variables, expected value and variance
- Solving geometry problems involving expected values
- Properties of expectation, such as linearity of expectation

Sample Problem:
(BMT-2012-Tournament-Round2-P1) 4 balls are distributed uniformly at random among 6 bins. What is the expected number of empty bins?

Chapter 9: Markov Chains

- Solving problems using Markov chains and state diagrams

Sample Problem:
(PUMaC-2010-Combinatorics-4) Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares and there are \$5 bills in the top right and bottom left squares, as shown below.



Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Chapter 10: Geometric Counting

- Solving counting problems related to geometric objects
- Euler's Formula

Sample Problem:
(AIME-2018-I-7) A right hexagonal prism has height 2. The bases are regular hexagons with side length 1. Any 3 of the 12 vertices determine a triangle. Find the number of these triangles that are isosceles (including equilateral triangles).

Chapter 11: Generating Functions

- Using generating functions to turn counting problems into algebra
- Counting number of partitions

Sample Problem:
(CHMMC-2010 Winter-Team-6) Zach rolls five tetrahedral dice, each of whose faces are labeled 1, 2, 3, and 4. Compute the probability that the sum of the values of the faces that the dice land on is divisible by 3.

Chapter 12: Catalan Numbers

- Using Catalan numbers to solve counting problems

Sample Problem:
(HMMT Feb-2007-Combinatorics-10) A subset S of the nonnegative integers is called *supported* if it contains 0, and $k + 8, k + 9 \in S$ for all $k \in S$. How many supported sets are there?