MC45A
AIME Advanced Algebra

Chapter 1: Word Problems

- Developing logical analysis and boost creative thinking by solving word problems.
- Converting word problems into mathematical equations and solving AIME level system of equations.

Sample Problem:
(HMMT Feb-2012-Guts-17) Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William's reverses direction at the same speed). Initially, Mark's orb is moving at 1/1000 meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

Chapter 2: Sequences & Series

- Finding patterns in sequences by looking at small cases.
- Using trig substitution and invariance in sequence problems.
- Understanding recurrence relations and solving linear recurrences.

- Finding closed-form formulas for sequences.

Sample Problem:
(AIME-2008-I-12) On a long straight stretch of one-way single-lane highway, cars all travel at the same speed and all obey the safety rule: the distance from the back of the car ahead to the front of the car behind is exactly one car length for each 15 kilometers per hour of speed or fraction thereof (Thus the front of a car traveling 52 kilometers per hour will be four car lengths behind the back of the car in front of it.) A photoelectric eye by the side of the road counts the number of cars that pass in one hour. Assuming that each car is 4 meters long and that the cars can travel at any speed, let M be the maximum whole number of cars that can pass the photoelectric eye in one hour. Find the quotient when M is divided by 10.

Chapter 3: Functions-1

- Solving equations that involve special functions such as floor, ceiling and absolute value
- Counting functions using information about its domain and range

Sample Problem:
(Mohammad Jafari) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x + f(y))) = 2x + f(x + y)$$

for all real x, y.

Chapter 4: Functions-2

- Solving functional equations using substitution, injectivity, and surjectivity, symmetry

Sample Problem:
(HMMT Nov-2015-Guts-26) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y. If $f(2) = 0$, compute $f(2015)$.

Chapter 5: Polynomials-1

- Finding roots of some cubic, quartic, and higher degree polynomials using substitution, binomial theorem
- Vieta's theorem and its applications
- Using techniques such as long division, factor theorem and rational root theorem when finding roots of higher degree polynomials

Sample Problem:
(PUMaC-2010-Algebra-5) Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots of f are given by α, β, and γ, find

$$\left(\frac{1}{\alpha - 2}\right)^2 + \left(\frac{1}{\beta - 2}\right)^2 + \left(\frac{1}{\gamma - 2}\right)^2.$$

Chapter 6: Polynomials-2

- Solving polynomial equations using Lagrange interpolation and Finite differences

Sample Problem:
(HMMT Feb-2010-Algebra-6) Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p.

Chapter 7: Logarithm

- Solving AIME level problems involving logarithms and natural logarithm

Sample Problem:
(AIME-2006-I-9) The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

Chapter 8: Trigonometry

- Solving algebra problems using trig substitution, trig identities and formulas

Sample Problem:
(SMT-2014-Algebra Tiebreaker-3) Compute $\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}}$.

Chapter 9: Complex Numbers-1

- Having a deep knowledge of complex numbers, finding roots of polynomials with complex roots
- Algebraic operations involving complex numbers and complex plane
- Problem solving techniques using Euler's formula and de Moivre's formula

Sample Problem:
(AIME-2013-I-14) For $\pi \leq \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \ldots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \ldots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

Chapter 10: Complex Numbers-2

- Finding roots of unity and using algebraic operations on roots of unity to solve problems

Sample Problem:
(HMMT Feb-2006-Guts-21) Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

Chapter 11: System of Equations

- Solving system of equations using polynomials, substitutions and symmetry

Sample Problem:
(PUMaC-2016-Combinatorics-3) Alice, Bob, Charlie, Diana, Emma, and Fred sit in a circle, in that order, and each roll a six-sided die. Each person looks at his or her own roll, and also looks at the roll of either the person to the right or to the left, deciding at random. Then, at the same time, Alice, Bob, Charlie, Diana, Emma and Fred each state the expected sum of the dice rolls based on the information they have. All six people say different numbers; in particular, Alice, Bob, Charlie, and Diana say 19, 22, 21, and 23, respectively. Compute the product of the dice rolls.

Chapter 12: Inequalities

- Finding minimum/maximum of algebraic expressions using elementary properties of inequalities, such as transitivity and algebraic operations on inequalities
- Arithmetic Mean - Geometric Mean (AM-GM) Inequality
- Cauchy-Schwarz Inequality
- Some advanced inequalities such as Rearrangement Inequality, Jensen's Inequality and weighted AM-GM Inequality

Sample Problem:
(SMT-2018-Team-8) Eddy has two blank cubes A and B and a marker. Eddy is allowed to draw a total of 36 dots on cubes A and B to turn them into dice, where each side has an equal probability of appearing, and each side has at least one dot on it. Eddy then rolls dice A twice and dice B once and computes the product of the three numbers. Given that Eddy draws dots on the two dice to maximize his expected product, what is his expected product?