MC40C
AIME Basic Counting

Chapter 1: Basic Counting Techniques

- Solving counting problems using techniques such as casework and complementary counting

Sample Problem:
(AIME-2008-I-7) Let S_i be the set of all integers n such that $100i \leq n < 100(i+1)$. For example, S_4 is the set $400, 401, 402, \ldots, 499$. How many of the sets $S_0, S_1, S_2, \ldots, S_{999}$ do not contain a perfect square?

Chapter 2: Counting Sets & PIE

- Solving counting problems using the Principle of Inclusion and Exclusion (PIE)

Sample Problem:
(Kevin Liu) How many functions $f : \{1, 2, \ldots, 6\} \to \{1, 2, \ldots, 6\}$ are there such that $\{1, 2, 3\}$ is a subset of the range of f?

Chapter 3: Path Counting & Bijections

- Solving counting problems using bijections
- Solving path-counting problems

Sample Problem:
(Brice Huang) How many ways are there to write 10 as the sum of any number of positive integers if different orderings of the same sum are distinguishable?

Chapter 4: Stars and Bars

- Solving counting problems using the Stars and Bars method

Sample Problem:
(PUMaC-2014-Team-5) How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a + b + c = 103$?

Chapter 5: Binomial

- Solving counting problems involving binomials and multinomials
- Binomial identities such as Hockey-Stick Identity and Vandermonde's Identity

Sample Problem:
(AIME-2000-II-7) Given that

$$\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!}$$

find the greatest integer that is less than $\frac{N}{100}$.

Chapter 6: Counting with Recursion

- Identifying which counting problems can be solved using recursions
- Finding and solving recursions

Sample Problem:
(PUMaC-2014-Combinatorics-4) Amy has a 2×10 puzzle grid which she can use 1×1 and 1×2 (1 vertical, 2 horizontal) tiles to cover. How many ways can she exactly cover the grid without any tiles overlapping and without rotating the tiles?

Chapter 7: Probability

- Solving difficult probability problems
- Conditional probability and Bayes' Theorem

- Geometric probability

Sample Problem:
(AIME-2014-II-6) Charles has two six-sided dice. One of the die is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$ and each of the other five sides has probability $\frac{1}{15}$. Charles chooses one of the two dice at random and rolls it three times. Given that the first two rolls are both sixes, the probability that the third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Chapter 8: Expected Value

- Random variables, expected value and variance
- Solving geometry problems involving expected values
- Properties of expectation, such as linearity of expectation

Sample Problem:
(HMMT Feb-2010-Guts-9) Indecisive Andy starts out at the midpoint of the 1-unit-long segment $\overline{HT}$. He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H, and if the coin is tails, he moves halfway towards endpoint T. After his 2010 moves, what is the expected distance between Andy and the midpoint of $\overline{HT}$? Express your answer in decimals.

Chapter 9: Markov Chains

- Solving problems using Markov chains and state diagrams

Sample Problem:
(PUMaC-2010-Combinatorics-4) Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares and there are \$5 bills in the top right and bottom left squares, as shown below.

\$1				\$5
	E			
\$5				\$1

Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Chapter 10: Geometric Counting

- Solving counting problems related to geometric objects
- Euler's Formula

Sample Problem:
(ARML-2014-Team-2) A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a+\sqrt{b}\pi$, where a and b are rational numbers. Compute (a,b).

Chapter 11: Generating Functions

- Using generating functions to turn counting problems into algebra
- Counting number of partitions

Sample Problem:
(Christopher Shao) Find the number of solutions to $a+b+c=4$ if $-3 \leq a \leq -1, 0 \leq b \leq 2, 3 \leq c \leq 5$ and a, b, and c are integers.

Chapter 12: Catalan Numbers

- Using Catalan numbers to solve counting problems

Sample Problem:
(HMMT Feb-2001-Guts-11) 12 points are placed around the circumference of a circle. How many ways are there to draw 6 non-intersecting chords joining these points in pairs?