MC40A
AIME Basic Algebra

Chapter 1: Word Problems

- Developing logical analysis and boost creative thinking by solving word problems.
- Converting word problems into mathematical equations and solving AIME level system of equations.

Sample Problem:
(PUMaC-2012-Team-2.3.1) For some reason, people in math problems like to paint houses. Alice can paint a house in one hour. Bob can paint a house in six hours. If they work together, it takes them seven hours to paint a house. You might be thinking "What? That's not right!" but I did not make a mistake.
When Alice and Bob work together, they get distracted very easily and simultaneously send text messages to each other. When they are texting, they are not getting any work done. When they are not texting, they are painting at their normal speeds (as if they were working alone). Carl, the owner of the house decides to check up on their work. He randomly picks a time during the seven hours. The probability that they are texting during that time can be written as r/s, where r and s are integers and $\gcd(r, s) = 1$. What is $r + s$?

Chapter 2: Sequences & Series

- Finding patterns in sequences by looking at small cases.
- Using trig substitution and invariance in sequence problems.

- Understanding recurrence relations and solving linear recurrences.
- Finding closed-form formulas for sequences.

Sample Problem:
(AMC12-2016-B25) The sequence (a_n) is defined recursively by $a_0 = 1, a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Chapter 3: Functions-1

- Solving equations that involve special functions such as floor, ceiling and absolute value
- Counting functions using information about its domain and range

Sample Problem:
(Hong Kong MC-2010-14) Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, e.g. $\lfloor \pi \rfloor = 3$. Given $f(0) = 0$ and

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n - 2\left\lfloor \frac{n}{2} \right\rfloor$$

for any positive integer n. If m is a positive integer not exceeding 2010, find the greatest possible value of $f(m)$.

Chapter 4: Functions-2

- Solving functional equations using substitution, injectivity, and surjectivity, symmetry

Sample Problem:
(Vishal Arul) Define $\{x\}$ to be the fractional part of x; that is, $\{x\} = x - \lfloor x \rfloor$. Define $a \circ b = \lfloor ab^2 + a^4(3-a)\{|b|^{1/a}\} + a^2 - 2b^2 - 5a \rfloor + 6$. What is $1 \circ (2 \circ (3 \circ \cdots (99 \circ 100) \cdots))$?

Chapter 5: Polynomials-1

- Finding roots of some cubic, quartic, and higher degree polynomials using substitution, binomial theorem
- Vieta's theorem and its applications
- Using techniques such as long division, factor theorem and rational root theorem when finding roots of higher degree polynomials

Sample Problem:
(Iurie Boreico) A rectangular box has volume equal to 6, surface area equal to 30, and diagonal equal to $\sqrt{34}$. The largest dimension of the box is $a+\sqrt{b}$ where a, b are positive integers. Find $a+b$.

Chapter 6: Polynomials-2

- Solving polynomial equations using Lagrange interpolation and Finite differences

Sample Problem:
(Jafar Jafarov) If a, b, c are roots of $x^3+2x+7=0$, find

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}$$

Express your answer as a common fraction in reduced form.

Chapter 7: Logarithm

- Solving AIME level problems involving logarithms and natural logarithm

Sample Problem:
(CHMMC-2010 Fall-Individual-4) Let

$$S=\log_2 9 \log_3 16 \log_4 25 \cdots \log_{999} 1000000.$$

Compute the greatest integer less than or equal to $\log_2 S$.

Chapter 8: Trigonometry

- Solving algebra problems using trig substitution, trig identities and formulas

Sample Problem:
(AMC12-2008-A25) A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ of points in the coordinate plane satisfies

$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n) \text{ for } n = 1, 2, 3, \ldots$$

Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

(A) $-\frac{1}{2^{97}}$ **(B)** $-\frac{1}{2^{99}}$ **(C)** 0 **(D)** $\frac{1}{2^{98}}$ **(E)** $\frac{1}{2^{96}}$

Chapter 9: Complex Numbers-1

- Having a deep knowledge of complex numbers, finding roots of polynomials with complex roots
- Algebraic operations involving complex numbers and complex plane
- Problem solving techniques using Euler's formula and de Moivre's formula

Sample Problem:
(AIME-2016-I-7) For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

Chapter 10: Complex Numbers-2

- Finding roots of unity and using algebraic operations on roots of unity to solve problems

Sample Problem:
(HMMT Feb-2008-Algebra-6) A root of unity is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

Chapter 11: System of Equations

- Solving system of equations using polynomials, substitutions and symmetry

Sample Problem:
(AIME-2000-I-7) Suppose that x, y, and z are three positive numbers that satisfy the equations $xyz = 1$, $x + \frac{1}{z} = 5$, and $y + \frac{1}{x} = 29$. Then $z + \frac{1}{y} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 12: Inequalities

- Finding minimum/maximum of algebraic expressions using elementary properties of inequalities, such as transitivity and algebraic operations on inequalities
- Arithmetic Mean - Geometric Mean (AM-GM) Inequality
- Cauchy-Schwarz Inequality
- Some advanced inequalities such as Rearrangement Inequality, Jensen's Inequality and weighted AM-GM Inequality

Sample Problem:
(SMT-2018-Algebra Tiebreaker-1) If a, b, c are real numbers with $a - b = 4$, find the maximum value of $ac + bc - c^2 - ab$.