MC35N
AMC 10/12 Advanced Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences
- Sum of squares and sum of cubes formula
- Sigma notation

Sample Problem:
(HMMT Nov-2013-Guts-9) Find the remainder when $1^2 + 3^2 + 5^2 + ... + 99^2$ is divided by 1000.

Chapter 2: Primes & Prime Factorization

- Definition of divisibility ($a \mid b$)
- Prime and composite numbers, Euclid's proof of the infinitude of primes
- Fundamental theorem of arithmetic
- Legendre's formula
- Prime number theorem, prime checking algorithms (optional)

Sample Problem:
(AIME-2006-II-3) Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

Chapter 3: Divisibility Rules

- Divisibility by numbers 2-11 inclusive
- Various problems involving divisibility, prime factorization, etc.

Sample Problem:
(Mehmet Kaysi) The number $\overline{406828a}$, where a is a digit, is an odd perfect square which is not a multiple of 9. What is the digit a?

Chapter 4: Number & Sum of Divisors

- General formula for the number and sum of divisors of a positive integer n, given its prime factorization
- Perfect, abundant, and deficient numbers (optional)

Sample Problem:
(ARML-2014-Individual-6) Compute the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.

Chapter 5: Factoring Techniques

- Difference of squares
- Sum and difference of cubes; sum and difference of n-th powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity
- Informal definition of an irreducible polynomial over the integers (e.g. x^2+y^2)

Sample Problem:
(SMT-2018-General-18) How many integer pairs (a, b) satisfy $\frac{1}{a}+\frac{1}{b}=\frac{1}{2018}$?

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Chapter 6: Number Bases

- Conversion between different number bases (emphasis on base 2, 8, 10, and 16)
- Arithmetic in different bases
- Fast base conversion (e.g. binary to hexadecimal)

Sample Problem:
(SMT-2012-Advanced Topics-2) Find the sum of all integers $x, x \geq 3$, such that

$$201020112012_x$$

(that is, 201020112012 interpreted as a base x number) is divisible by $x-1$.

Chapter 7: GCD & LCM

- Definition of relatively prime
- Computing the GCD and LCM using the prime factorization
- Computing the GCD of two numbers using the Euclidean algorithm

Sample Problem:
(HMMT Nov-2015-Guts-15) Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution."

Chapter 8: Modular Arithmetic

- Basic properties of the modulo (reflexive, symmetric, transitive, etc.)
- Proof of divisibility rules using modular arithmetic
- Modular inverses
- More advanced modulo calculations involving basic operations

Sample Problem:
(AIME-2010-I-2) Find the remainder when $9 \cdot 99 \cdot 999 \cdot \cdots \cdot \underbrace{99\cdots 9}_{\text{999 9's}}$ is divided by 1000.

Chapter 9: Fermat's Little Theorem

- Definition of reduced residue systems $\pmod m$
- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem: (SMT-2019-Discrete-1) How many nonnegative integers less than 2019 are not solutions to $x^8 + 4x^6 - x^2 + 3 \equiv 0 \pmod 7$?

Chapter 10: Euler Theorem

- Definition of the totient function $\phi(n)$
- Using the totient function on basic problems involving relatively prime integers
- Definition of Euler's totient theorem
- Demonstrating that Fermat's little theorem is a special case of Euler's totient theorem

Sample Problem:
(Ata Pir) Find the smallest integer n, such that $\frac{\varphi(n)}{n} < \frac{1}{4}$.

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to more advanced modular arithmetic problems
- Directly computing solutions to systems of congruences
- Using the Chinese remainder theorem backwards

Sample Problem:
(AMC10-2010-A24) The number obtained from the last two nonzero digits of 90! is equal to n. What is n?

(A) 12 **(B)** 32 **(C)** 48 **(D)** 52 **(E)** 68

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Bézout's identity, using the reverse Euclidean Algorithm
- Chicken McNugget theorem
- Finding Pythagorean triples
- Using modular arithmetic to solve Diophantine equations, or to show there is no integer solution

Sample Problem:
(CHMMC-2010 Winter-Individual-9) Let A and B be points in the plane such that $AB = 30$. A circle with integer radius passes through A and B. A point C is constructed on the circle such that $\overline{AC}$ is a diameter of the circle. Compute all possible radii of the circle such that BC is a positive integer.