MC25N
AMC 8/MathCounts Advanced Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences (e.g. sum of the first n positive integers)
- Sum of the first n perfect squares, cubes

Sample Problem:
(Richard Spence) The sum $1^3 + 2^3 + 3^3 + \ldots + n^3$ is equal to a perfect fourth power, where $n \geq 1$. What is the smallest possible value of n?

Chapter 2: Primes & Prime Factorization

- Definition of divisibility
- Fundamental Theorem of Arithmetic
- Determining if a number is prime or not
- Legendre's formula

Sample Problem:
(AMC10-2002-B6) For how many positive integers n is $n^2 - 3n + 2$ a prime number?

(A) none **(B)** one **(C)** two **(D)** more than two, but finitely many **(E)** infinitely many

Chapter 3: Divisibility Rules

- Divisibility rules for all positive integers up to and including 11

Sample Problem:
(Sean Shi) What is the smallest five-digit positive integer divisible by 5 with digit sum 27?

Chapter 4: Number of Divisors

- Determining the number of divisors of a positive integer n using the prime factorization of n
- Multiplicative functions

Sample Problem:
(Tiancheng Qin) Given that b and n are both positive integers at most 15, what is the greatest number of divisors that b^n can have?

Chapter 5: Sum of Divisors

- Definition of $\sigma(n)$
- Determining the sum of divisors of a number n using the prime factorization of n

Sample Problem:
(Ali Gurel) Find the smallest two consecutive squares whose sum of divisors are the same.

Chapter 6: Factoring Techniques

- Difference of squares
- Simon's Favorite Factoring Trick (SFFT)
- Sum of cubes, difference of cubes
- Sophie-Germain identity

Sample Problem:
(Ali Gurel) Find the sum of prime divisors of 4891.

Chapter 7: Number Bases

- Representing numbers in different bases
- Converting numbers between bases (emphasis on base 2, 8, and 16)
- Arithmetic in different bases

Sample Problem:
(Nathan Zhang) Find the base-10 value of $11_2 + 22_3 + 33_4 + \ldots + 99_{10}$.

Chapter 8: GCD & LCM

- Computing the GCD and LCM of two or more numbers
- Euclidean algorithm
- Relation between gcd and lcm ($lcm(a, b) = ab/gcd(a, b)$)

Sample Problem:
(Ali Gurel) How many pairs of ordered positive integers (a, b) are there such that $\text{lcm}(a, b) = 48$ and $\gcd(a, b) = 4$?

Chapter 9: Modular Arithmetic

- Introduction to the congruence operator ($a \equiv b \pmod{m}$)
- Basic properties of modulo (reflexive, symmetric, transitive)
- Computing remainders by finding patterns
- Proof of the divisibility rules for 3, 9, and 11

Sample Problem:
(Kevin Chang) The Fibonacci sequence is the sequence $1, 1, 2, 3, 5, \ldots$, where each term after the second is the sum of the previous two terms. What is the units digit of the 10^6th Fibonacci number?

Chapter 10: Fermat's Little Theorem

- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:
(Richard Spence) What is the units digit of $1^{12} + 2^{12} + 3^{12} + \ldots + 2019^{12}$?

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to basic modular arithmetic problems
- Solving basic systems of congruences

Sample Problem:
(Kevin Chang) How many integers between 1 and 2520, inclusive, are divisible by 36, but not by 5 or 7?

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Chicken McNugget theorem
- Bézout's identity
- Pythagorean triples
- Using modular arithmetic to show that a Diophantine equation has no solutions

Sample Problem:
(Richard Spence) Hexagonal-shaped tubing is sold in packages of 7 and 19 tubes. What is the smallest number k such that for any $n \geq k$, I can always buy exactly n tubes?